Marking a Physical Sphere with a Projected Platonic Solid

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Abstract

Although mathematicians often use coordinates for the vertices of the Platonic solids, a physical sphere is not a priori situated within this same coordinate system. In this paper, we describe how to locate points relative to one another on the surface of a sphere, in order to mark vertices and edges of each of the spherically projected Platonic solids without first coordinatizing. While certain methods for the cube and octahedron are standard and the tetrahedron method is known in the temari community, and is clear to any mathematician, in this paper, one procedure that leads to all three is given as a necessary first step to finding the dodecahedron. The author believes the dodecahedron and icosahedron procedures to be original to Western language scholarship. While some steps are theoretically exact, others only approximate; however at one step the approximation can be made as exact as one desires. Furthermore, the procedures allow for making adjustments to account for the fact that physical balls always fail to be perfect spheres.

1. Art---Motivation and Applications

Humans are drawn to the spherical form and to symmetrical division. What could be more natural than artists wishing to divide spheres into congruent regions for the purpose of decoration? Indeed, this occurs across media with wood, ceramics, painting, and embroidery to name but a few. This author can imagine many other applications. For examples of existing artwork incorporating exact subdivision of spheres, see Figure 1.

Figure 1: *From left to right---Roads Untaken, by George Hart, wood tiled fiberglass sphere [3]; Sphere of Influence, by Richard Weber, carved ceramic spherical rattle [11]; Which Way, by Dick Termes, painted sphere [7]; temari ball with cube/octahedral symmetry, by the author.*

Only one of the pieces in Figure 1 was divided into a projected Platonic solid. Yet Hart's piece is fundamentally based upon a truncated Platonic solid. See [4] for his discussion of the piece. Termes' piece comes from taking a projected icosahedron, dividing each triangle into three subtriangles by connecting the incenter with the vertices, and then forming diamonds by pairing triangles across the original trianglar boundaries. Auxiliary triangles have been drawn that obscure this construction. To learn about his multiperspective technique, consult [8]. Weber's ceramic sphere relies only on sectors, but one can imagine him using Platonic projection to inform future designs, given the large variety of divisions he has already employed. See [11].

This author frequently works within the medium of wrapped and embroidered Japanese thread balls, called temari balls. The inspiration for this paper came from the mathematical impurity of the standard method for dividing the sphere for a projected dodecahedral/icosahedral subdivision, known as a C10 subdivision in the temari literature. Hence, the goal was to find a simultaneously theoretically more exact and technically purer method for making this subdivision. The techniques used here are derived from the traditional Japanese techniques of measuring with paper tape, and directions are given as though pins can be pushed into the sphere, as if it were made of Styrofoam. If this is not possible, such as if the sphere is made of glass or metal, perhaps the pinheads can be affixed to the sphere pointing outwards with glue. However, with smooth media it may be easiest to use only one pin (the North Pole on which the paper tape swivels) and to mark all other pinpoints and edges with pencil or other erasable marker. Nonetheless, any artist interested in dividing his or her spherical region into those given by a projected Platonic solid will be able to do so by following the essence of these directions.

2. Geometric Applications---Math in the Wild

Imagine holding in your hand a ball, or better yet, go get a ball---a super ball, tennis ball, Styrofoam ball, metal ball, or any other ball. Notice that what you have is round, but not a perfect sphere. Observe that finding the diameter is not immediate. Finding the circumference is much more natural, but this too poses a challenge. How will you know when you have the circumference rather than something close? What will you use to find the circumference? A rubber band? A piece of string? A paper tape? Using our methods, we will not need to measure distances per se, but will use only relative distances. Therefore, we will not need a ruler. However, we will need to be able to record distances for future reference, such as for division. Therefore, a rubber band will not work for our purposes. The string will only work if it has no elasticity. A paper tape, a thin strip of paper, will work best, because it has no elasticity, and we will be able to record our observations directly onto our measuring device with a pen or pencil.

In what follows, we describe a procedure to divide a ball into sections defined by the central projection of an inscribed Platonic solid onto the surface of the ball. With paper tape, origami techniques, two colors of pins (black and white), a pen, and thread, we can mark the surface of a ball with the vertices and edges of any Platonic solid. We make no attempt to use the known coordinates of the Platonic solids or to situate the ball within a coordinate system.

3. Finding the Circumference of the Sphere

How do we determine the circumference of the given sphere? Near one end of a paper tape, place a black pin through the paper tape and into the ball. Wrap the tape around the sphere, attempting to form a great circle, and make a fold in the tape where the paper meets the pin. If the paper really constitutes a great circle at this point, then this is the longest the tape will be and still lie flat around the ball. If not, then when the loop of tape is swiveled around the ball, there will be some other position at which the tape will need to be longer in order to reach the pin while lying flat around the ball. So, to check that the paper is the right length from pin to fold, turn the ball around the pin, holding the paper tape fixed and reflattening the tape as the ball is turned. If the tape continues to just reach the pin exactly, then the initial circumference estimate was correct. However, if the length of tape undershoots the pin at some points as the ball is turned, then make sure to straighten the tape and refold it to lengthen the distance to the fold so that the new distance just reaches the pin. Remember that because we want a great circle, the tape will always be lengthened, never shortened.

4. Marking a North/South Pole Pair on the Sphere

All of the Platonic solids except for the tetrahedron have antipodal vertex pairs. Therefore, we wish to mark one such pair. We may rename the pin already placed through the paper tape (used to find the circumference of the ball) the North Pole. The antipodal point to this is the South Pole, which is halfway around the ball. To find this new point, fold the distance on the paper tape measuring the circumference in half, and then we can easily measure the distance from the North Pole to the South Pole. Swiveling the tape any direction is valid, as all great circles should have the same length. Place a black pin, not through the paper tape, at the South Pole, as shown in Figure 2. Because the ball is not perfectly spherical, prudence suggests checking pin placement by measuring the half circumference from the North Pole by swiveling the ball and holding onto the paper tape, as before. The South Pole pin may have to be moved slightly a few times until it is in a suitable spot from all directions, that is, it is close to being equidistant from the North Pole measured in all directions. This adjustment may be necessary be due to thickness of the paper obfuscating initial pin placement as much as ball defect.

Figure 2: *Fold the circumference length in half, flatten the tape against the sphere, and place the South Pole pin at the center end of the tape.*

5. Marking a Projected Cube, Octahedron, or Tetrahedron

In the temari literature, the projected octahedron marking, known as C6 is commonly marked separately from the projected cube marking, known as C8. However, because our ultimate goal is to mark the projected dodecahedron, for which we will need to know the cube side length, we will not mark these separately here. Rather we refer the reader to standard sources such as [1], [2], [6], [9], [10], and [13]. Nevertheless, to begin the construction process, consider the six vertices of the octahedron shown in Figure 3: there are three antipodal pairs, one of which we have already placed. The other two pairs, made up of four points, we call equator points.

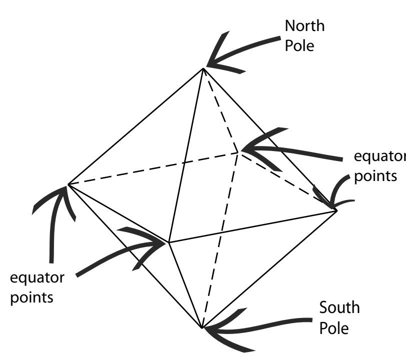


Figure 3: *Octahedron with vertices marked as pole or equator points.*

To place equator points, first fold the circumference tape (which should still be securely anchored at the North Pole) into fourths, and measure one half distance down the ball, placing eight to ten pins approximately equally spaced around the equator of the ball---that is, on the great circle lying equidistant between the North and South poles. See Figure 4(a). Now make a small cut through one half of the paper tape to the pin at the North Pole allowing removal of the paper tape without requiring disturbance of the pin placement. Divide the circumference length into eights by folding. This will be most accurate if halving folds are made one at a time rather than allowing paper thickness to build up. Using the equator pins as a guideline, place the paper tape around the equator, abutting each pin from the same side of the ball, with circumference ends matching up, as in Figure 4(b).

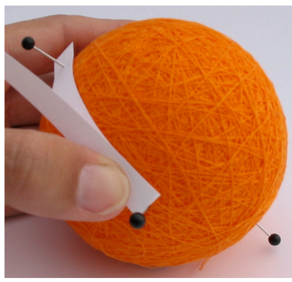
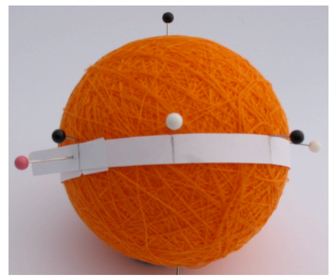
(a)  (b) 

Figure 4: *(a) Place equator pins by measuring one fourth of the distance down from the North Pole. (b) After placing a number of equator pins, use these arbitrarily spaced pins as guides to place the circumference tape around the middle of the sphere, like a belt. This tape, divided into eighths, will be used to repositions the pins equally around the equator of the sphere.*

Place a pin at every fold mark, alternating colors (black and white), simultaneously removing the pins used simply to mark the equator. (It is helpful to simply move the pins that do not correspond to a fold mark to a point rather than introducing a whole new set of pins.) We now have ten pins on the ball---the six black vertices correspond to the six vertices of the octahedron, and the four white vertices further divide the equator.

Next choose a pair of opposite black vertices on the equator and reconceptualize the ball so that those are the poles. Use the four pins already lying on the equator with respect to the new poles to place the paper tape around the new equator. Keeping those pins in place, use the paper tape to place four more (white) pins on the equator, dividing it into eighths. Finally, locate the unique pair of (black) antipodal pins on the initial equator that can be used as poles so that the equator corresponding to this pair will be perpendicular to both of the other equators. Then repeat the procedure of putting in four additional (white) pins to divide this third equator into eighths.

Both the vertices and the edges of the cube can now be marked by strategically wrapping thread around the ball, guided by the pins. For each pole pair, one at a time, tightly wrap all four great circles passing through both poles and one pair of opposing equator points. See Figure 5(a). From the perspective of the pole, doing so divides the ball into eight equal sectors, looking something like an orange. As this division is completed for the three pole pairs, the sector lines cross. Indeed, at eight points on the ball, three sector lines cross at the point, as exhibited in Figure 5(b). These points are the vertices of the cube. They may be marked with a pin, if desired. Note also that the points are already joined by sectors, which form the edges of the cube. Hence, the cube has been completed, and the octahedron has been completed, as well.

(a)  (b) 

Figure 5: *(a) Four great circles pass through each octahedral vertex guided by the marking pins. (b) As a result, three great circles meet in the center of each octahedral triangle to form the vertices of the cube.*

To make a tetrahedron, note that by taking alternating upper and lower vertices of a cube you have an inscribed tetrahedron, and this remains true when projected onto the sphere. Simply join the vertices to obtain the edges.

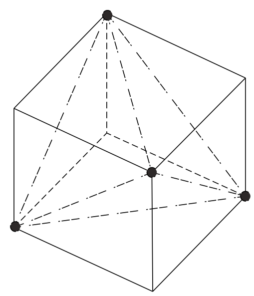
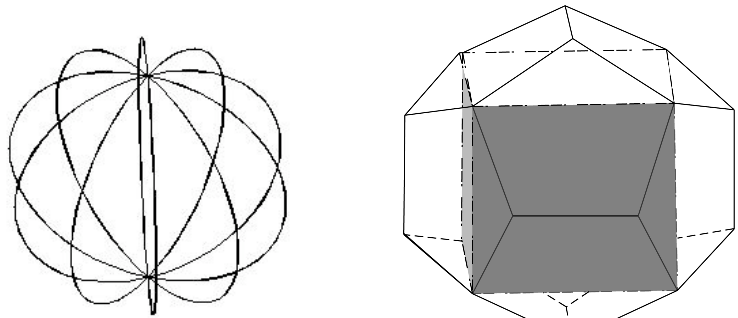


Figure 6: *Tetrahedron lying in cube. Six tetrahedral edges are marked with irregular dotted lines and four tetrahedral vertices are marked with small circles.*

6. Marking a Projected Dodecahedron or Icosahedron

To make a projected dodecahedron or icosahedron is slightly more difficult. On your existing paper tape, mark off the side length of the cube you have made. The paper tape is now spent in terms of folding, but it contains important distance information. So cut a new tape, longer than the ball circumference, marking the new tape with the circumference length approximately one centimeter from one of the ends. We now divide the circumference into tenths, which can be done in several ways. Use the Fujimoto approximation technique, an origami technique beautifully described by Hull in [3], to divide the circumference into fifths. Repetition of the technique allows for as much accuracy as one desires. Or, use a ruler and compass construction to divide the length into fifths. After satisfactory division into fifths, use folding to further subdivide the circumference into tenths. Choose an existing pole pair and use its equator pins to accurately wrap the newly divided paper tape circumference around the equator. Repin the equator so that pins mark each tenths subdivision. Remove all pins other than these twelve (poles and equator) and all marking threads. Now wind threads to mark five new great circles through the five pairs of antipodal points on the equator and passing through the poles. This divides the ball into ten equal sectors from the perspective of the poles, as in Figure 7(a).



(a) (b)

Figure 7: (a) *Five equally spaced great circles placed around an invisible sphere; (b) The solid cube, two grey faces and one white, inscribed in the dodecahedron. The fact that one edge of the cube spans two edges of the dodecahedral pentagon is key to the construction.*

The two poles represent centers of two projected pentagonal faces of the dodecahedron. The other ten centers of the faces of the dodecahedron will lie on the sector lines, but they alternate up and down some distance from the equator. Thus knowing the distance between two adjacent projected centers is essential to marking the remainder of the ball.

Now choose any one of the sector lines and go up near a pole. Using your paper tape, marked off with half of the edge of the cube length, measure orthogonally over from your latitude line until your length hits the latitude line two over, as shown in the middle of Figure 9. Do so by moving the straight folded edge of the paper tape nearer or farther away from the pole on the latitude line, not by changing the angle of the tape! Note that the ninety degree angle here is local, and originates from the interaction of the folded edge of the paper tape with the straight guideline (marking thread). Mark the intersection point of the mark on the tape with the latitude line two over with a pin. See Figure 10 (a).

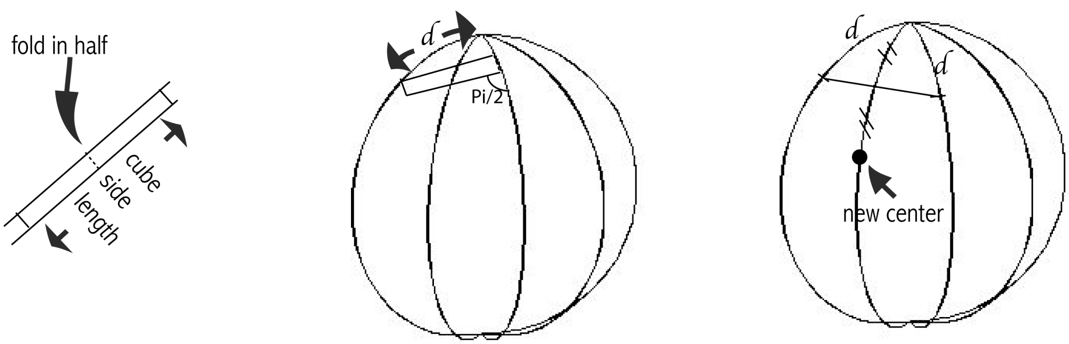


Figure 9: *Diagrams for steps involved in locating adjacent centers on the dodecahedron.*

(a) (b) (c)

Figure 10: *(a) Using the half cube side length tape; (b) Determining distance d; (c) Marking off a new distance d.*

If the most recent step has been done accurately, the new pin is at one of the vertices of the pentagon surrounding the pole. Now mark the distance from the pole to this pin on a piece of paper tape and call the distance *d*, as depicted in Figure 10(b). Mark the distance *d* with a pin on a second latitude line two over from the one already marked off by the pin. This is illustrated in Figure 10(c). This marks off a second vertex. Use these two vertex pins and either a piece of paper tape or a piece of string pulled tight to mark off (with a pin) the intersection of the line segment they generate, which is an edge of the dodecahedron, with the latitude line between them, shown on the right side of Figure 9. Now measure the distance between the pole (one center of the dodecahedron) and this latest pin. This distance is the projected perpendicular distance between the center of a pentagonal face and one of its edges. Then double that distance along that same latitude line and place another pin, also shown on the right side of Figure 9. This new pin represents the center of an adjacent face on the dodecahedron. Now that we have solved the problem of finding the distance between adjacent centers on the projected dodecahedron, we have theoretically completed our solution.

Practitioners may like to have the steps for marking the dodecahedron and icosahedron spelled out; therefore, we do so below. Begin by measuring and copying the distance between two adjacent centers onto a piece of paper tape. Mark this distance from the North Pole on every other sector line. Next mark the same distance, but this time from the South Pole, on the remaining sector lines. Now all centers have been marked onto the ball. See Figure 11(a). Checking that the distance between the centers is indeed constant can ensure accuracy. Ten of the centers do not have five great circles emanating from them, namely the ten just placed on the sector lines. However, for symmetry of the ball, and to mark edges, these great circles should be marked with thread. Choose any one of these ten centers and its antipodal pin. One of the great circles is already present---the sector line upon which the pin lies. The other four great circles will be guided by the eight remaining centers, divided into antipodal pairs. Continue for each of the other four pairs, noticing that fewer and fewer great circles are needed at each next pin pair to complete the five great circle set. When each center has 10 sector lines equally dividing the ball from the perspective of the center, the entire ball will appear to be divided up into twelve pentagons. This is the projection of the dodecahedron. These twelve pentagons will be themselves divided up into 10 triangles. Alternately, the ball could be seen as being divided into 20 triangles, which are subdivided into six subtriangles. This is the icosahedron, as depicted in Figure 11(b).

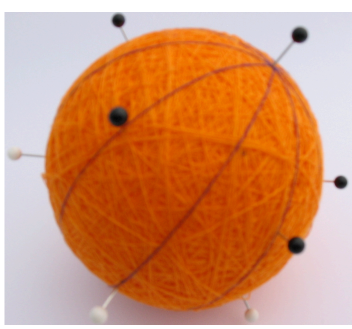
(a)  (b) 

Figure 11: *(a) Pins on spindle lines repositioned; (b) Completely marked projected dodecahedron and icosahedron.*

7. Notes on the Procedures

The methods in this paper extend those given in the temari sources available to the author, including [1], [2], [6], [9], [10], [13], and [14], and are analogous in many ways to straight edge and compass constructions rather than ruler use. The author's goal was to find a way to make a C10 subdivision using "rules" of temari, which she took to mean measuring with paper tape and using traditional paper folding techniques. As a mathematician, she was also prepared to use straightedge and compass techniques. She did not, however, allow herself rulers or protractors. This eliminated the usual techniques for C10 subdivisions. The methods for the octahedron and cube are standard and available in any book on making temari balls, such as [2], [6], [10], and [13] as well as on temari websites, the most comprehensive of which are [1] and [9], and in [14]. Less common, but obvious to any mathematician, is the method given for the tetrahedron, which was included in [14]. The methods for the first three Platonic solids were collapsed here for brevity, but included as a necessary step to making the projected dodecahedron/icosahedron. Via the method in this paper the most multifaceted of the Platonic dual pairs is marked simultaneously, but the edge length of the cube projected onto the same sphere is needed in order to complete the construction.

The existing temari literature contains two other techniques for marking the projected dodecahedron/icosahedron. The first accepted technique is based on using the radius times an approximation of the radian measure of the $\arccos\left(\frac{1}{\sqrt{5}}\right)$ as the distance between centers. The second ordinary technique involves using a v-ruler (a special tool combining a protractor constantly set at seventy-two degrees with metric rulers on its sides) and an approximation of side length from a chart, which essentially does the same as the first approximation, though possibly more accurately. As much as possible, the method set forth in this paper is akin to a classical construction—it may not always be the fastest, but it exhibits a purity of form.

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