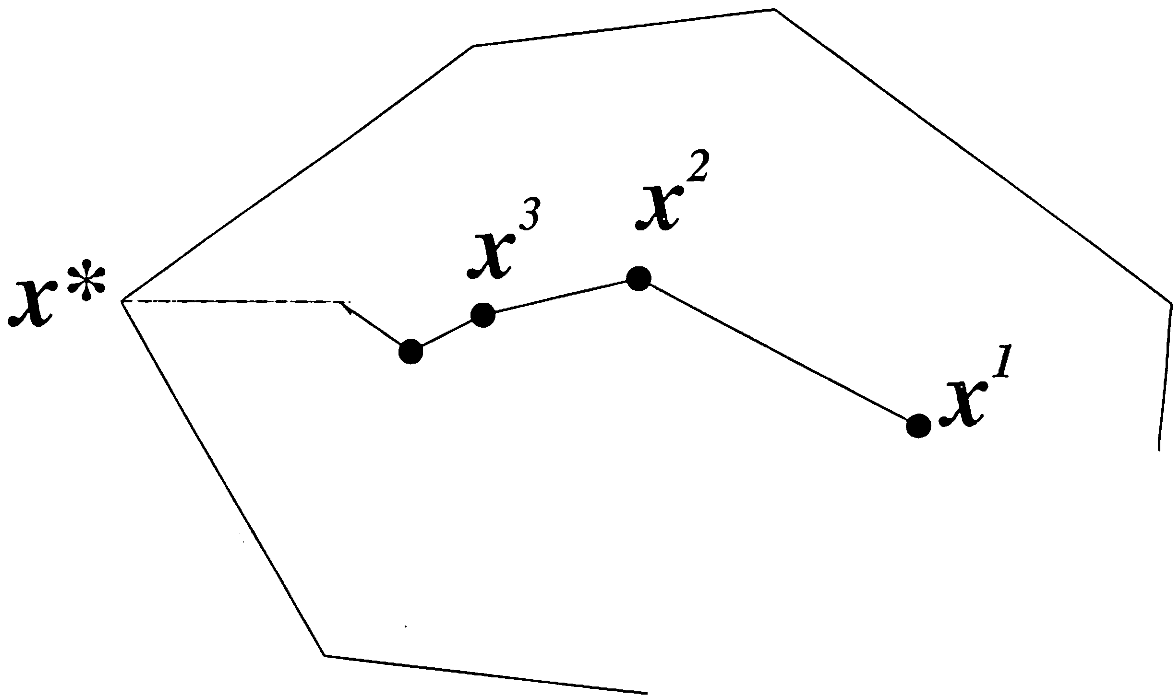
Chpt 7

HW 8
Read Chpt 7.1–7.2
Ex: 5.1, 5.3
7.2 (a), (c)
(*) Prove that the properties (T1) ~ (T6) holds for T_k defined on pg 147.
19 Nov.

□ Interior Point Method

- Basic Idea:



Step 1: Start with an interior solution.

Step 2: If current solution is good enough, STOP.
Otherwise,

Step 3: Check all directions for improvement and move to a better interior solution.
Go to Step 2.

□ Interior Movement

- Format:

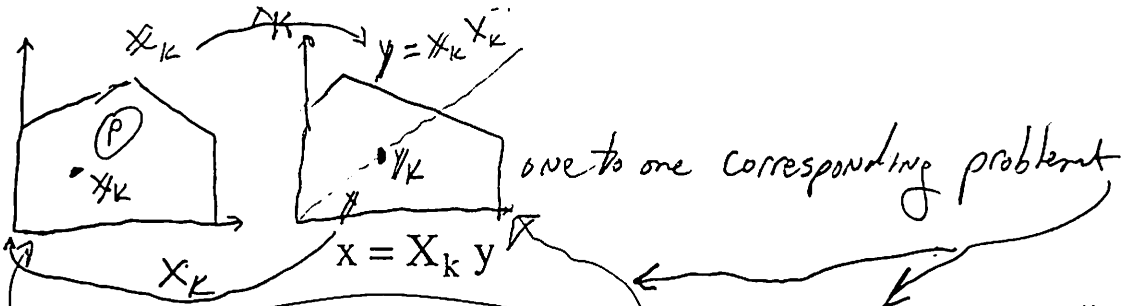
$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha \mathbf{d}_{\mathbf{x}}^k$$

$$\begin{cases} \alpha \geq 0 : & \text{Step} - \text{length} \\ \mathbf{d}_{\mathbf{x}}^k \in R^n : & \text{moving direction} \end{cases}$$

- Questions:

1 How to find a "good" direction $\mathbf{d}_{\mathbf{x}}^k$?

2 How far should we move?



Min $c^T x$	Min $c^T X_k y$
s.t. $Ax = b$	s.t. $A X_k y = b$
$x \geqslant 0$	$y \geqslant 0$

MIN $c^T X_k y = c'y$

$A X_k = \begin{vmatrix} a_{11}, a_{12} \dots a_{1n} \\ a_{21}, a_{22} \dots a_{2n} \\ \vdots \\ a_{m1}, a_{m2} \dots a_{mn} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{vmatrix}$

$A' = a_{11}x_1^k, a_{12}x_2^k, \dots, a$

$\mathbf{x}^k$ $\qquad$ $\mathbf{y}^k = e$

old coordinate, step length, unit vector

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \alpha_k \frac{\mathbf{d}_{\mathbf{y}}^k}{||\mathbf{d}_{\mathbf{y}}^k||}$$

$d_y^k = (I - A'^T(A'A'^T)^{-1}A')(-$

$= (I - (AX_k)^T(AX_kX_k^TA)$

$$\mathbf{d}_{\mathbf{y}}^k = [I - X_k\mathbf{A}^T(\mathbf{A}X_k^2\mathbf{A}^T)^{-1}\mathbf{A}X_k] \cdot(-X_k\mathbf{c})$$

$= (I - X_kA^T(AX_k^2A^T)^{-1}AX_k)(-X_kc$

$$\begin{aligned} \mathbf{x}^{k+1} &= X_k\mathbf{y}^{k+1} \\ &= X_k\mathbf{y}^k + \alpha_k X_k \frac{\mathbf{d}_{\mathbf{y}}^k}{||\mathbf{d}_{\mathbf{y}}^k||} \\ &= \mathbf{x}^k + \frac{\alpha_k}{||\mathbf{d}_{\mathbf{y}}^k||}\mathbf{d}_{\mathbf{x}}^k \end{aligned}$$

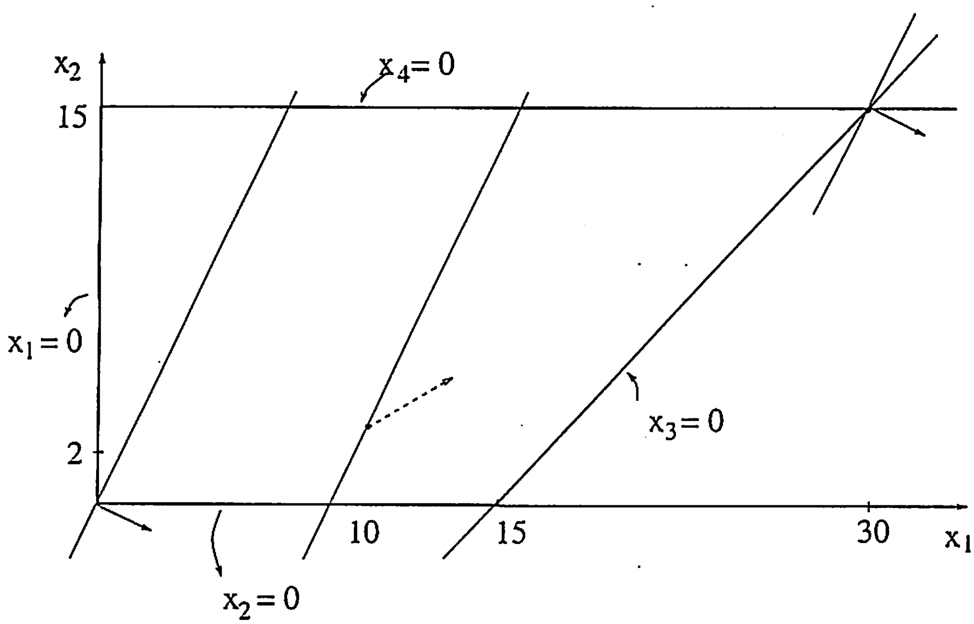
$X_k \mathbf{d}_{\mathbf{y}}^k = d_x^k$

$$\therefore \ \underline{\mathbf{d}_{\mathbf{x}}^k = -X_k[I - X_k\mathbf{A}^T(\mathbf{A}X_k^2\mathbf{A}^T)^{-1}\mathbf{A}X_k]X_k\mathbf{c}}$$

$\underline{\alpha_k = 0.99 \text{ (say)}}$ $\quad 0 < \alpha_k < 1$

AN EXAMPLE

$$
\begin{aligned}
\min \quad & -2x_1 + x_2 \\
\text{s.t.} \quad & x_1 - x_2 \leq 15 \\
& x_2 \leq 15 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Reformulate to standard form

$$\begin{array}{lrl} \min & -2x_1 + x_2 & \\ \text{s.t.} & x_1 - x_2 + x_3 & = 15 \\ & x_2 + x_4 & = 15 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

and $\mathbf{x}^0 = \begin{pmatrix} 10 \\ 2 \\ 7 \\ 13 \end{pmatrix}$ is feasible

MATRIX FORMAT

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} > \mathbf{0} \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$

$$\mathbf{c} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}^0 = \begin{bmatrix} 10 \\ 2 \\ 7 \\ 13 \end{bmatrix}$$

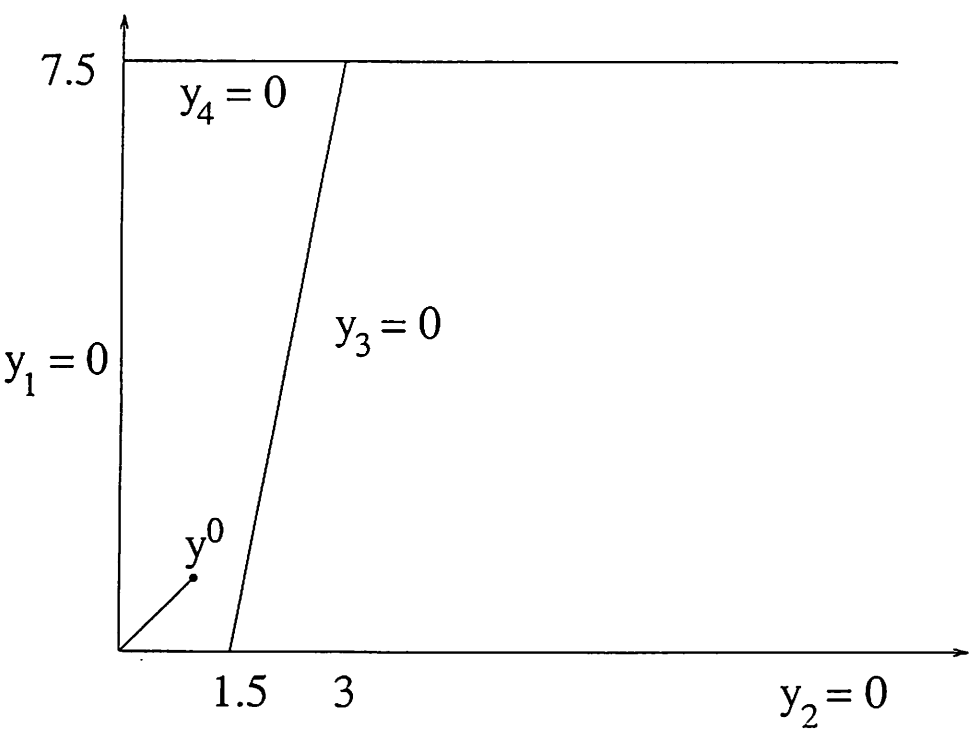
$$\mathbf{X}_0 = \begin{bmatrix} 10 & & & 0 \\ & 2 & & \\ 0 & & 7 & \\ & & & 13 \end{bmatrix}$$

SCALING

$$\mathbf{y} = X_0^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{10} & & & \\ & \frac{1}{2} & & 0 \\ 0 & & \frac{1}{7} & \\ & & & \frac{1}{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \begin{aligned} y_1 &= x_1/10 = 1 \\ y_2 &= x_2/2 = 1 \\ y_3 &= x_3/7 = 1 \\ y_4 &= x_4/13 = 1 \end{aligned}$$

The problem is transformed to

$$\begin{array}{lllr} \min & -2(10y_1) + (2y_2) & & = -20y_1 + 2y_2 \\ \text{s.t.} & 10y_1 - 2y_2 & +7y_3 & = 15 \\ & 2y_2 & +13y_4 & = 15 \\ & y_1, y_2, y_3, y_4 & & \geq 0 \end{array}$$



The new matrix form

$$\begin{aligned} \min \quad & \bar{\mathbf{c}}^T \mathbf{y} \\ \text{s.t.} \quad & \bar{\mathbf{A}}\mathbf{y} = \mathbf{b} \\ & \mathbf{y} > \mathbf{0} \end{aligned}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} 10 & -2 & 7 & 0 \\ 0 & 2 & 0 & 13 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$$

$$\bar{\mathbf{c}} = \begin{bmatrix} -20 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \ \text{and } \mathbf{y}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} \tilde{A} = AX_0 \\ \tilde{\mathbf{c}} = X_0 \mathbf{c} \\ \mathbf{y}^0 = X_0^{-1} \mathbf{x}_0 \end{cases}$$