

Autocorrelation and Causality

(Subtlety of FFT calculations)

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Abstract

The Wiener Khinchin theorem is widely recognized as the basis for calculating autocorrelations by means of the Fourier Transform. The need to account for issues of causality when doing such calculations is not well known. This article shows how to avoid errors of numerical type when computing the autocorrelation by means of the FFT.

Background

Relative to the Wiener Khinchin theorem, the autocorrelation of $E(t)$ is defined in a pure mathematics sense as:

$$C(t) = \int_{-\infty}^{\infty} \bar{E}(\tau) E(t + \tau) d\tau$$

where E can be expressed in terms of its Fourier components

$$E(\tau) = \int_{-\infty}^{\infty} E_{\nu} e^{-2\pi i \nu \tau} d\nu$$

so that its complex conjugate is given by

$$\bar{E}(\tau) = \int_{-\infty}^{\infty} \bar{E}_{\nu} e^{2\pi i \nu \tau} d\nu$$

Substituting $\bar{E}(\tau)$ and $E(t + \tau)$ into the definition of $C(t)$ gives

$$\begin{aligned} C(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \bar{E}_{\nu} e^{2\pi i \nu \tau} d\nu \right] \left[\int_{-\infty}^{\infty} E_{\nu'} e^{-2\pi i \nu' (t+\tau)} d\nu' \right] d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{E}_{\nu} E_{\nu'} e^{-2\pi i \tau (\nu' - \nu)} e^{-2\pi i \nu' t} d\tau d\nu d\nu' \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{E}_{\nu} E_{\nu'} \delta(\nu' - \nu) e^{-2\pi i \nu' t} d\nu d\nu' \end{aligned}$$

$$C(t) = \int_{-\infty}^{\infty} \bar{E}_{\nu} E_{\nu} e^{-2\pi i \nu t} d\nu \quad (1)$$

So it is seen that the autocorrelation can be calculated by a sequential calculation of two Fourier

transforms as follows: (i) first take the transform of $E(t)$, (ii) then generate the 'intermediate' that is the square of that transform by multiplying each Fourier component by its complex conjugate, and finally (iii) obtain the autocorrelation by taking the transform of the 'intermediate'.

When dealing with a digitized time record comprising N measured values, distributed over a total interval of time τ , each of the components of the autocorrelation can be numerically calculated in terms of the measured Y_k having the mean value $\langle Y \rangle$ as follows:

$$c_k = \frac{\sum_{i=1}^{N-k+1} (Y_i - \langle Y \rangle) (Y_{i+k-1} - \langle Y \rangle)}{\sum_{i=1}^N (Y_i - \langle Y \rangle)^2} \quad (2)$$

In Eq. (2) the digital values of the (implicitly specified) time are given by $t_k = k \delta t$ where $\delta t = \tau/N$.

Numerical calculations of the Fourier Transform have become a routine part of scientific analyses, following the invention of the 'fast' version by Cooley and Tukey [1]. A preference thus exists for the calculation of autocorrelations on the basis of Eq. (1) rather than the use of Eq. (2). If one is careful to construct a (column) 'vector of values' (VoV) on which the FFT algorithm operates, then the two methods yield the same result. It is very easy, however, to fall into the trap that is next described.

Example to illustrate the cause for a common erroneous calculation

Shown in Fig. 1 is a square wave of finite duration having a 'period' of 8 s and sampled at a rate of 1 Hz.

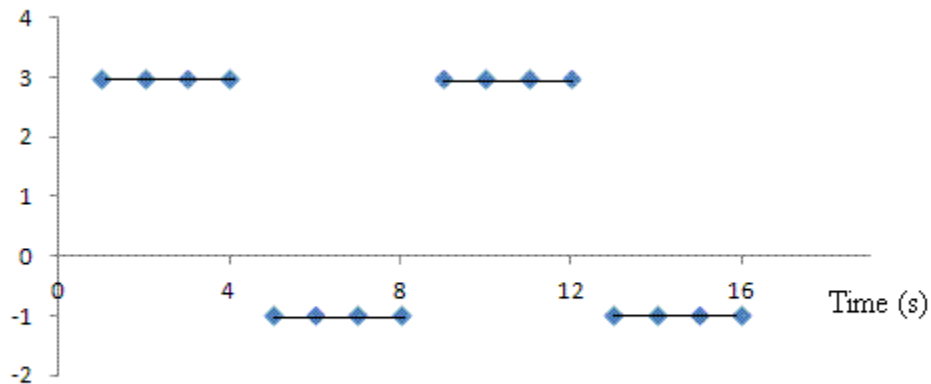


Figure 1. Illustration of the record for which we want the autocorrelation.

On the basis of Eq.(1), it is probably natural for you to try and solve the problem as follows:

- (1) Take the FFT (of a VoV corresponding to the record), using a total number of points $N = 16$.
- (2) Multiply your result by its complex conjugate.
- (3) Take the FFT of your product to obtain (what you expect to be) the autocorrelation.

Using the Excel spreadsheet, these three steps were taken. Before calculations, the offset of the record was removed and it was rescaled to have an amplitude of 1. For proper transform normalization, each of the (real) components of the final transform were divided by the square of 16. The result is shown in Fig. 3.

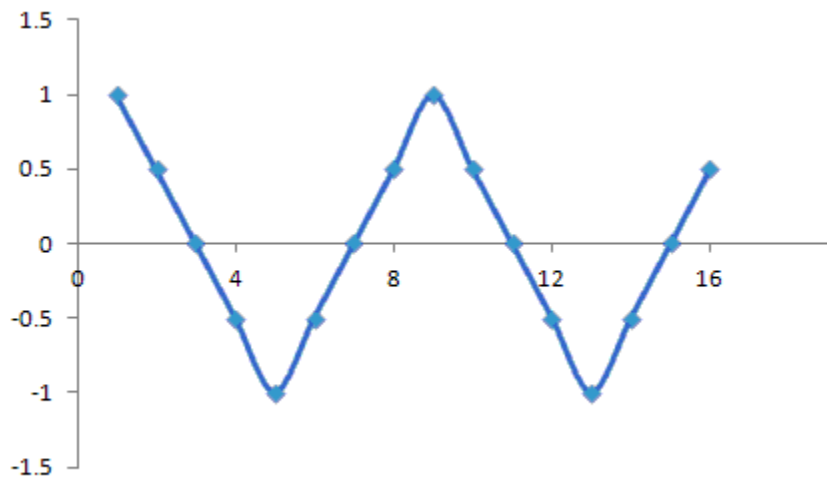


Figure 2. Erroneous autocorrelation resulting from the FFT operating on a VoV that is completely filled by the record values.

Correction for the error

The problem with the preceding method derives from the failure to account for causality. Correction is straightforward by the following means:

- (1) Double the size of the VoV and set every one of its values initially to zero.
- (2) Then anywhere you should choose within the VoV, replace the zero's there with the record values.
- (3) Thereafter do the calculations indicated previously, but in the final result, use only the first half of the output.

Doing as before, using the Fourier Analysis (ToolPak add-in), but now operating with a 32-pt FFT on a VoV for which the upper half was 'padded' with zero's, Fig. 3 was generated.

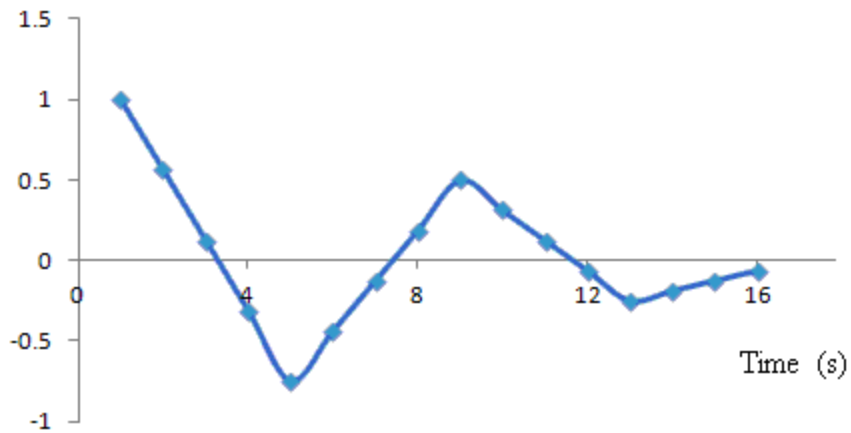


Figure 3. Proper autocorrelation resulting from the FFT operating on a (column) VoV that was half filled by the record values and 'padded' with zero's in the other half.

The autocorrelation was also calculated using Eq. (2), and the result was found to be identical with the values plotted in Fig. 3.

Reference

[1] <http://www.ams.org/journals/mcom/1965-19-090/S0025-5718-1965-0178586-1/S0025-5718-1965-0178586-1.pdf>