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Tutorial on Gravitational Pendulum Theory Applied to Seismic Sensing of Translation and Rotation

by Randall D. Peters

Abstract Following a treatment of the simple pendulum provided in Appendix A, a rigorous derivation is given first for the response of an idealized rigid compound pendulum to external accelerations distributed through a broad range of frequencies. It is afterward shown that the same pendulum can be an effective sensor of rotation, if the axis is positioned close to the center of mass.

Introduction

When treating pendulum motions involving a noninertial (accelerated) reference frame, physicists rarely consider the dynamics of anything other than a simple pendulum. Seismologists are concerned, however, with both instruments more complicated than the simple pendulum and how such instruments behave when their framework experiences acceleration in the form of either translation or rotation. Thus, I look at the idealized compound pendulum as the simplest approximation to mechanical system dynamics of relevance to seismology. As compared to a simple pendulum, the properties of a compound pendulum can be radically modified according to the location of its axis relative to the center of mass.

Pendulum Theory

The Simple Pendulum

The theory of the simple (mathematical) pendulum is provided in Appendix A.

Theory of an Exemplary Compound Pendulum

When an external force is applied to an extended object whose shape is invariant, it generally causes two responses: a linear acceleration of the center of mass and a rotational acceleration around the center of mass. For this system, Newton's laws of translation and rotation are applied respectively to each response. The coupled-equation sets obtained from these two forms of the law are then combined to obtain the single equation of motion. This method will be used to analyze the compound pendulum described next.

As noted, we are concerned with small displacements, where the drive acceleration is rarely large enough to generate amplitudes in excess of 1 mrad. Thus, the nonlinear influence of the $\sin\theta$ term is inconsequential; that is, the instrument is nearly isochronous ($\sin\theta \approx \theta$ and $\cos\theta \approx 1$).

Figure 1 shows a pendulum similar to various instruments of importance in seismology. It is a true pendulum in the sense that restoration is due to the gravitational field of the Earth at its surface, little **g**. Some other instruments common in physics and sometimes labeled pendulums do not employ a restore-to-equilibrium torque based on the Earth's field. For example, restoration in the Michell–Cavendish balance that is used to measure big G (Newtonian universal gravitational constant) is provided by the elastic twist of a fiber (TEL-Atomic, Inc., 2008). It is sometimes called a torsion pendulum. Many seismic instruments are also called pendulums, even though restoration may be mostly provided by a spring. Two commonly employed spring types are the LaCoste zero-length and the astatic. As discussed in Appendix B, the rotation response at low frequencies of a spring-restored oscillator is significantly different from that of a gravity-restored pendulum.

While the fiber of the Cavendish balance is secured only at the top, other torsion pendulums use a vertical fiber that is also secured at both ends. The best known example from seismology is the Wood–Anderson seismograph, used by Richter to define the original earthquake magnitude scale. By means of an adjustable period, a similar instrument can be configured to operate with large tilt sensitivity (Peters, 1990). As with any long-period mechanical oscillator, the maximum period (and the maximum sensitivity) of the tiltmeter is regulated by the integrity of its fiber spring. Acceptable stability against spring creep is difficult to achieve when the period is greater than about 30 sec (de Silva, 2007). A common seismology instrument for which the challenge to long-period stability is well known is the garden-gate horizontal pendulum. Rodgers (1968) recognized its potential for a variety of measurements.

Although the instrument in Figure 1 is idealized in the form of a two-element compound pendulum, it nevertheless is useful for illustrating a variety of important properties. The primary idealization is the assumption of a rigid structure. For reasons of material creep, and the placement of mass M_1 above the axis, real pendulums of this type experience structural deformation. In the absence of integrity sufficient

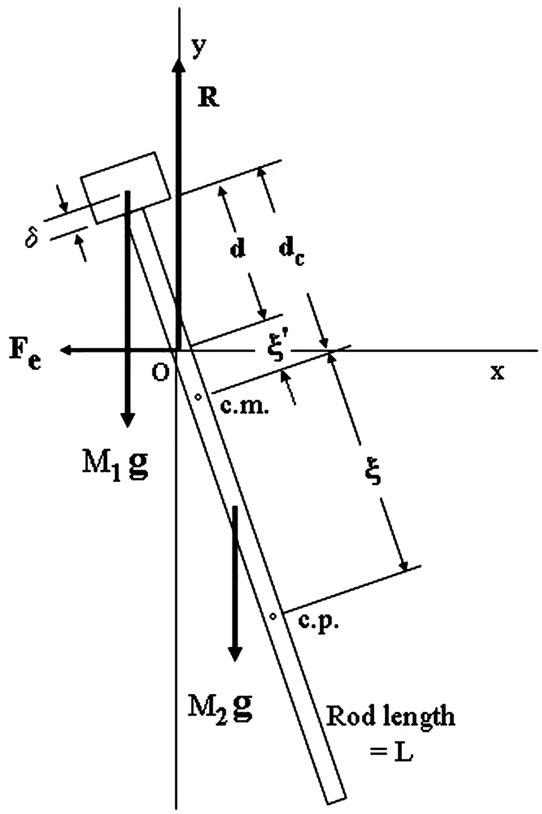

Figure 1. A compound pendulum used to illustrate some properties of interest to seismometry. Force vectors are shown that act on both the pendulum and its axis. c.m stands for center of mass; c.p. stands for center of percussion.

to maintain an essentially rigid structure, the center of mass of M_1 migrates slowly toward ever lower states of potential energy. Even though the rate of this deformation may be low, the rod will, through secondary creep, continue to bend about the axis. Upon initial setup and until primary creep has run its course, the equilibrium position of such an instrument is not fixed, not even for the highest Q materials used in fabrication.

The defect structures responsible for creep also result in hysteretic internal-friction damping. Ignore for the moment the force F_e. As the pendulum oscillates between positive and negative displacements, the three forces $\boldsymbol{R}$, $M_1\mathbf{g}$, and $M_2\mathbf{g}$ are responsible for a periodic flexural strain of the pendulum arm and pivot. We will not consider here the detailed physics whereby this strain lags behind an associated stress in the rod. Peters (2005a) shows how the Q of the system is naturally proportional to the square of the eigenfrequency. This proportionality results from the flexural interplay between the restoring $M_2\mathbf{g}$ and the destoring $M_1\mathbf{g}$. This property also allows one to tune the oscillator to a long period, by increasing M_1 or δ, and thus causing the center of mass to approach the axis. Subsequently, I show that this method can be used to convert the pendulum from an instrument more sensitive to translational acceleration into one that is more sensitive to rotation about O.

Newton's second law requires that the vector sum of the forces external to the pendulum must be equal to the total mass $M = M_1 + M_2$ times the acceleration of the center of mass a_{cm}. Additionally, the vector sum of external torques acting on the center of mass is equal to the moment of inertia about that point, I_{cm}, times the angular acceleration, $d^2\theta/dt^2$.

The axis at O is stationary in the accelerated frame of reference in which the instrument is located. This axis is accelerating in the negative x direction because of the horizontal force F_e acting through the axis. This external force and the reaction force $\boldsymbol{R}$ are input from the ground via the case that supports the pendulum. The force $\boldsymbol{R}$ balances the weight of the pendulum when it is at equilibrium with $F_e = 0$. Because we are concerned with motions for which $\theta \ll 1$, we quickly identify the first of several expressions used to develop the equation of motion: $(M_1 + M_2)g \approx R$.

For various calculations, a convenient reference point is the top of the uniform rod of mass M_2 and length L. Unlike the rod, M_1 that is placed above the axis is dense enough to be approximated as a point mass. For the purpose of torque calculations, the total mass of the pendulum is concentrated at the center of mass, whose position below the reference position at the top of the rod is indicated as d_c in Figure 1. Newton's law for translational acceleration of the center of mass and small rotation about the center of mass yields

$$-F_e \approx Ma_{\mathrm{cm}}, \qquad M = M_1 + M_2 \tag{1}$$

and

$$(d_c - d)(F_e - R\theta) \approx I_{\mathrm{cm}}\ddot{\theta}. \tag{2}$$

An additional relationship required to obtain the equation of motion without damping involves the transformation from the center of mass frame to the axis frame,

$$a_{\mathrm{cm}} \approx (d_c - d)\ddot{\theta} + a(t), \tag{3}$$

where $a(t) = a_{\mathrm{axis}}$ is the acceleration of the ground responsible for the pendulum's motion.

Combining equations (1) to (3) with $(M_1 + M_2)g = R$ one obtains

$$[I_{\mathrm{cm}} + M(d_c - d)^2]\ddot{\theta} + (d_c - d)Mg\theta \approx -(d_c - d)Ma(t). \tag{4}$$

Because of the parallel-axis theorem (Becker, 1954), the term in brackets multiplying the angular acceleration is recognized to be the moment of inertia I about axis O.

Up to this point I have ignored the effects of frictional damping. In the conventional manner a linear damping term is added to equation (4) to obtain the following linear approximation for the equation of motion of this compound pendulum:

$$\ddot{\theta}+\frac{\omega_0}{Q}\dot{\theta}+\omega_0^2\theta\approx-\frac{\omega_0^2}{g}a(t),\qquad \omega_0^2=\frac{M(d_c-d)g}{I},$$
$$d_c=\frac{M_2\frac{L}{2}-M_1\delta}{M}. \tag{5}$$

The equation is seen to be identical to that of the driven simple pendulum (equation A3, given in Appendix A, except with a drive term added to the right-hand side, and the expression for its eigenfrequency is considerably more complicated).

Radius of Gyration

It is common practice in physics and engineering to express the moment of inertia of a rigid object in terms of a length ρ called the radius of gyration and defined by

$$I=M\rho^2. \tag{6}$$

In keeping with this convention, we find for the compound pendulum just described

$$\rho^2=\rho_{\mathrm{cm}}^2+L^2\left(\frac{d_c}{L}-\frac{d}{L}\right)^2,$$
$$\rho_{\mathrm{cm}}^2=\frac{L^2}{M}\left\{M_2\left[\frac{1}{3}-\frac{d_c}{L}+\left(\frac{d_c}{L}\right)^2\right]+M_1\left(\frac{d_c}{L}+\frac{\delta}{L}\right)^2\right\}. \tag{7}$$

The distance $\xi'=(d_c-d)$ of axis O above the center of mass has a conjugate point called the center of percussion that is located a distance ξ below the center of mass; these points obey the relation

$$\xi'\xi=\rho_{\mathrm{cm}}^2. \tag{8}$$

As demonstrated in textbooks of mechanics (e.g., Becker, 1954, pp. 213), an impulsive force delivered perpendicular to the equilibrated pendulum at the center of percussion imparts no motion to the axis. An example of this phenomenon is the sweet spot of a baseball bat. When the ball is hit at that point there is no counter force to the batter's hands. For drive frequencies higher than the eigenfrequency, a sensor located at the center of percussion measures pendulum displacement equal in magnitude and π out of phase with the displacement of sinusoidal ground motion.

Determining the center of percussion allows one to define an effective (i.e., an equivalent simple pendulum) length by means of the following relationship:

$$\xi+\xi'=L_{\mathrm{eff}}, \tag{9}$$

where L_{eff} corresponds to the length of a simple pendulum having the same period as the compound pendulum. For the present compound pendulum

$$\frac{L_{\mathrm{eff}}}{L}=\left\{\left(\frac{d_c}{L}-\frac{d}{L}\right)^2+\frac{M_2}{M}\left[\frac{1}{3}-\frac{d_c}{L}+\left(\frac{d_c}{L}\right)^2\right]\right.$$
$$\left.+\frac{M_1}{M}\left(\frac{d_c}{L}+\frac{d}{L}\right)^2\right\}\Big/\left(\frac{d_c}{L}-\frac{d}{L}\right). \tag{10}$$

It should be noted from equation (10) that $d=d_c$ cannot be exceeded. Displacement greater than this value places the axis below the center of mass, resulting in an unstable inverted pendulum. In turn, it should also be obvious that the center of percussion moves below the bottom end of the rod, once a critical value of d has been exceeded. In the case where M_1 equals zero, a rod-only pendulum, instability occurs when $d>L/2$. Various properties of such a rod-only pendulum with $L=1$ m are shown in Figure 2.

The plots of Figure 3 are of similar type to those of Figure 2 but using an M_1 whose mass and placement yield the advantages described in Appendix C

Some Pendulum Characteristics

Appendix C describes some of the characteristics of the compound pendulum treated previously, including: (1) advantages of a pendulum with $M_1\neq 0$, (2) frequency dependence of the sensitivity to drive acceleration at low- and high-frequency limits, and (3) properties of the center of oscillation. The center of oscillation should not be confused with the center of percussion; only at high-excitation frequencies are the two points identical.

Transfer Functions

I now describe the transfer function of various pendulums, a continuous curve joining the two limiting frequency cases discussed previouisly. One is concerned with the steady-state response of the pendulum as described by complex exponentials. The result for the simple pendulum was given without derivation in equations (A7) and (A8) of Appendix A.

I now obtain the analogous results for the compound pendulum of Figure 1. The functions are derived using the Steinmetz phasor approach involving complex exponentials (Newburgh, 2004). The right-hand side of the equation of motion, equation (5), is written as $(\omega_0^2/g)a_{0,\mathrm{ground}}e^{j\omega t}$, where $j=(-1)^{1/2}$. For the entrained pendulum at steady state, its frequency is also ω and after taking the derivatives of the left-hand side of equation (5), we obtain the following expression, after dividing through by the common term $e^{j\omega t}$:

$$-\omega^2\theta_0+j\frac{\omega\omega_0}{Q}+\omega_0^2\theta_0=-\frac{\omega_0^2}{g}a_{0,\mathrm{ground}}. \tag{11}$$

By rearranging equation (11) and solving for the magnitude and the phase, one obtains

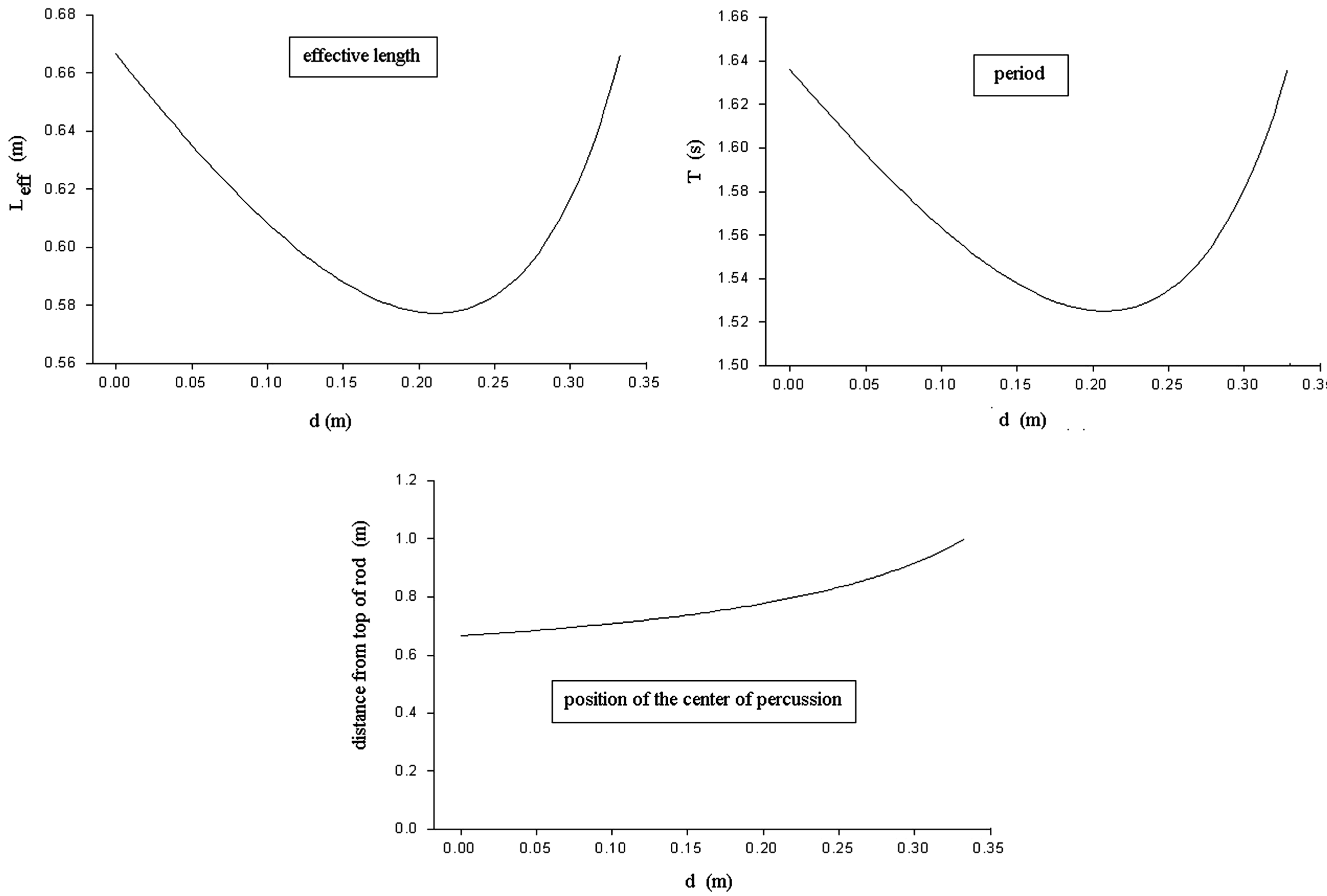

Figure 2. Properties of a meter-stick (rod) simple-compound pendulum. The graphs were generated with equation (10) using $M_1 = 0$ and $L = 1$.

$$\left|\frac{g\theta_0}{a_{0,\text{ground}}}\right| = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega_0^2\omega^2/Q^2}},$$
$$\phi = \tan^{-1}\frac{\omega_0\omega}{Q(\omega_0^2-\omega^2)}, \qquad \omega_0^2 = \frac{g}{L_{\text{eff}}}. \tag{12}$$

The phase term ϕ in equation (12) is plotted in Appendix A for two different values of Q. The pendulum and the drive are in phase when the drive frequency is much lower than the natural frequency. At the high-frequency extreme, the pendulum lags behind the drive by π.

The magnitude of the transfer function shown in equation (12) is appropriate to an instrument that measures the angular displacement of the pendulum. For an instrument that measures translational displacement of a point on the pendulum at some fixed distance from the axis, the equation is readily modified. For example, placement at the bottom of the rod yields

$$T_{f,\text{accel}} = \left(1-\frac{d}{L}\right)\frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega_0^2\omega^2/Q^2}}. \tag{13}$$

Displacement Transfer Function

The displacement transfer function of a compound pendulum is generated from the relationship between acceleration and displacement, that is, $a = -\omega^2 A$:

$$T_{f,\text{displ}} = \frac{L}{L_{\text{eff}}}\left(1-\frac{d}{L}\right)\frac{\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega_0^2\omega^2/Q^2}}, \tag{14}$$

where the displacement implied by equation (14) is the horizontal displacement of the bottom end of the compound pendulum relative to the instrument case. The transfer functions expressed by equations (13) and (14) are plotted in Figure 4 for the special case of the compound rod pendulum.

Pendulum Measurement of Tilt and Rotation

The response of the compound pendulum to rotation is treated in Appendix B. This appendix provides quantitative support for the claim that the pendulum of Figure 1 can be made sensitive to rotation while simultaneously insensitive to translational acceleration. As seen from the upper right-hand plot of Figure 3, moving the axis closer to the center of mass can readily cause the pendulum's effective length to become more than 30 times greater than the actual length of

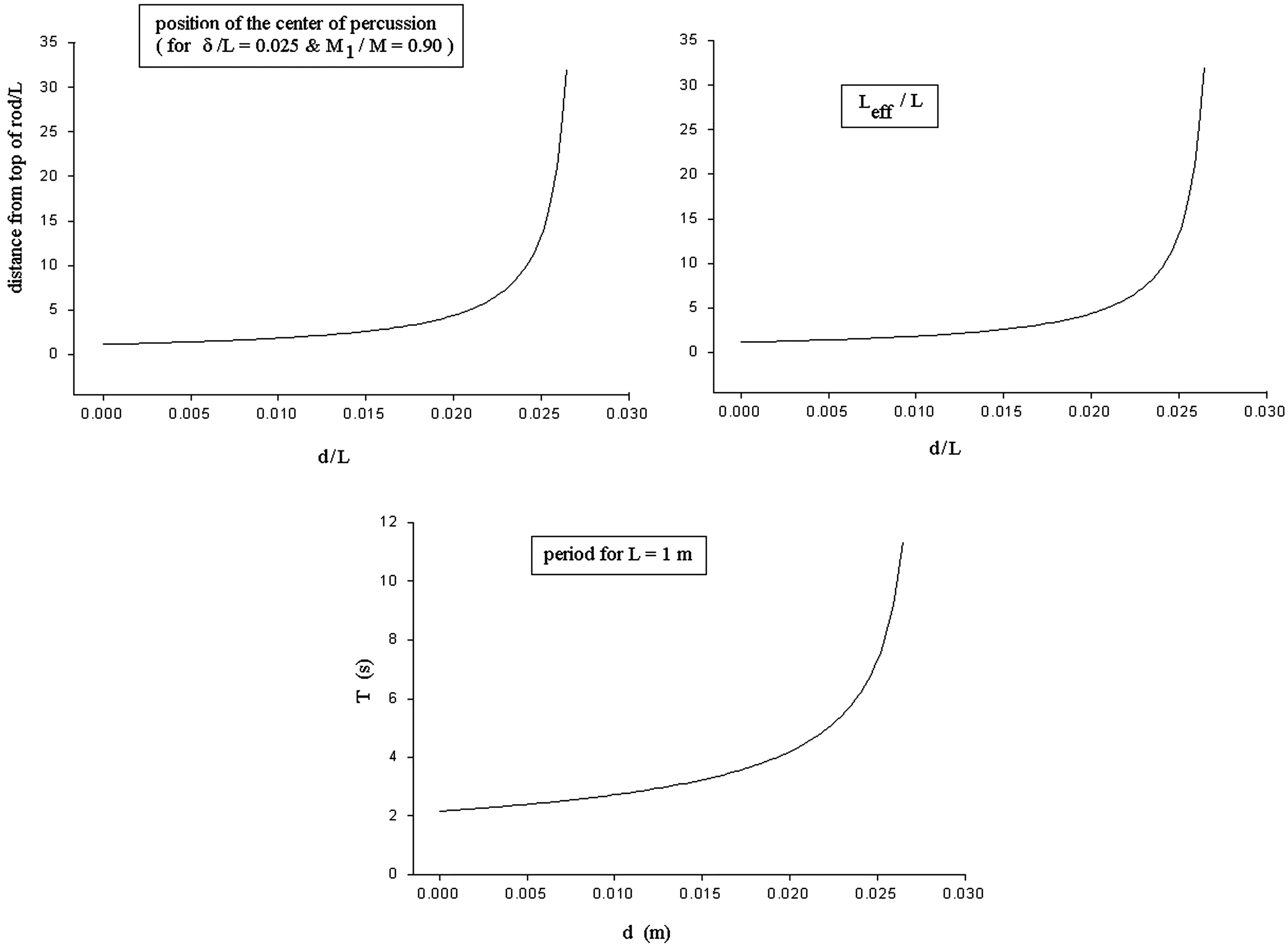

Figure 3. Properties of the two-component compound pendulum of Figure 1 for the parameters as indicated. For no value of d/L is the center of percussion located on the body of the pendulum.

the pendulum. In the process, equation (14) shows that the displacement response derived from translational acceleration diminishes. Because the rotational response is independent of the effective length, the relative response can become quite large. Figure 5 shows a 50-fold reduction in translational sensitivity for this case in which the center of mass is located 1 mm below the axis.

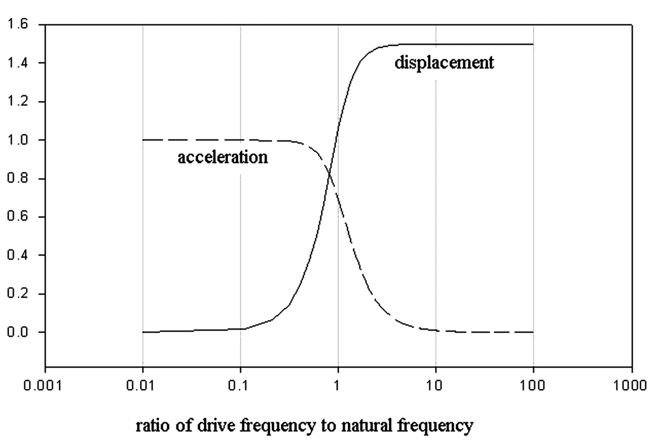

Figure 4. Acceleration and displacement transfer functions based on equations (13) and (14) using $d = 0$, $M_1 = 0$, and $Q = 0.7$.

Unconventional and Exotic Pendulums

Unconventional or exotic pendulums, such as the ones described in Appendix D, create additional complexities and useful phenomena.

Conclusions

Depending on geometry, compound pendulums can display many different types of responses. The single case treated previously, an idealized compound pendulum, provides a quantitative example. In addition to their diversity, gravity-restored pendulums can be remarkably simple, compared to many instruments widely used in seismology.

I have spent many years conducting pendulum research. In the course of two decades of intense study, I have come to the following conclusion—the classical pendulum should not be viewed as a relic. With modern digital technology,

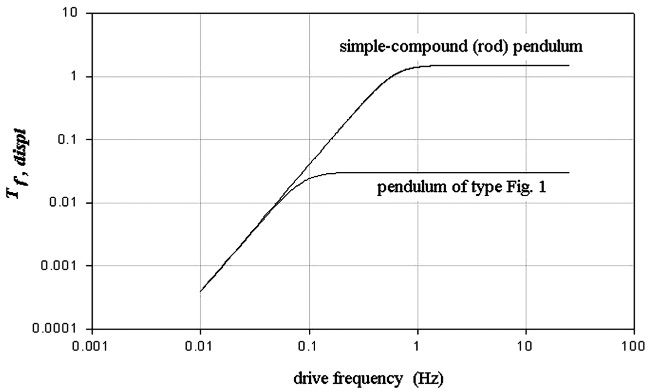

Figure 5. Comparison of the displacement transfer functions of a 1 m long simple-compound (rod) pendulum with the axis at one end and a pendulum of the type shown in Figure 1 with $L = 1$ m. For the latter $\delta = 2.500$ cm, $M_1 = 9.00M_2$, $d = 2.646$ cm, $d_c = 2.750$ cm, $L_{\mathrm{eff}} = 32.00$ m, and $T = 11.35$ sec. For both pendulums, $Q = 0.707$.

the pendulum has provided insights into several old foundational problems, such as the internal friction of metals, as well as issues more directly applicable to seismometry. Consideration of classical pendulums was extensive in the early days of seismometry, and modern seismologists would do well to take another look because improvements in technology have opened up new possibilities. For a variety of applications (including Earth rotation studies) the advantages of its low cost and high performance make this solution viable.

Data and Resources

No data were used in this article. Figures were made with Microsoft Excel for plots and Microsoft Paint for drawings.

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Appendix A

The Simple Pendulum

The simple pendulum (sometimes called a mathematical pendulum) is a convenient starting point for the consideration of pendulums in general. A simple pendulum comprises an inextensible, massless string of length L that supports a point mass m on one of its ends (Fig. A1). The other end of this string is the pendulum axis, a fixed stationary point whenever the instrument is located in an inertial coordinate frame. In addition to the only external force of significance (mg in Fig. A1), a tension force T is also shown. It has no component with which to influence the pendulum's motion; however, it must at all times balance the component of the bob weight along the direction of the string (unit vector $\mathbf{r}$). As the pendulum swings, this tension force varies periodically and would cause a variation in L if the string were extensible. The reaction to this tension force, acting at the top of the string, also would cause periodic sway in the case of any real pendulum. This sway can lengthen the period of oscillation and also give rise to hysteretic damping.

This idealized instrument is not achievable in practice, but the system is pedagogically useful and constitutes a useful benchmark against which real systems can be compared and numerically evaluated.

First we discuss the homogeneous equation of a simple pendulum's motion and then what happens to the dynamics of the pendulum when its axis experiences external accelerations.

Newton's second law is the basis for treating both the simple pendulum and the other instruments considered later. This law is best known to students in the special-case form $\mathbf{F} = m\mathbf{a}$, where the vector $\mathbf{F}$ is the sum of all external forces acting on scalar point mass m experiencing linear acceleration vector $\mathbf{a}$ as the result of $\mathbf{F}$. The acceleration thus calculated is valid only if m is constant. Newton actually provided this law in its general form, which says that the time rate of change of the vector momentum of an object is equal to the net external vector force acting on the object. This reduces to $\mathbf{F} = m\mathbf{a}$ for constant m.

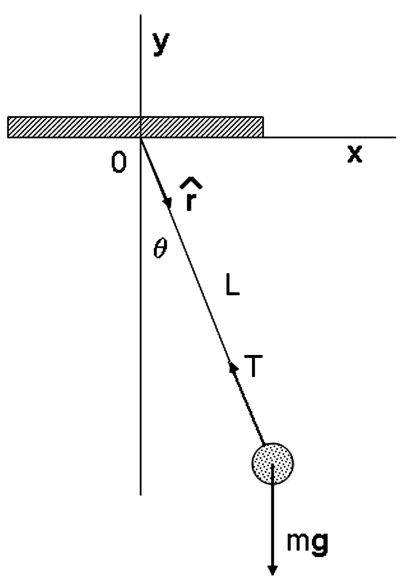

Figure A1. Geometry of the simple pendulum.

For systems like a symmetric pendulum that experiences rotation, the angular vector acceleration $\boldsymbol{\alpha}$ is related to the net vector torque $\boldsymbol{\tau}$ by way of Newton's second law of rotation in the form $\boldsymbol{\tau} = I\boldsymbol{\alpha}$. The scalar moment of inertia I is specified relative to the axis, which in Figure A1 is the origin O. Because the string is massless, $I = mL^2$. The magnitude of the angular acceleration is the second time derivative of θ or $d^2\theta/dt^2$. In the equations that follow, Newton's time derivative convention is followed, namely that one dot over the variable designates the first time derivative and two dots, the second time derivative. As with $\mathbf{F} = m\mathbf{a}$, $\boldsymbol{\tau} = I\boldsymbol{\alpha}$ is a special case. The general form of his rotational law says that the time rate of change of vector angular momentum equals the net external applied vector torque.

Because it is not an extended body in the usual sense, a simple pendulum is very easily treated using Newton's second law in the form of rotation. To begin with, following initialization at $\theta_0 \neq 0$, let the pendulum be swinging in free decay in an inertial coordinate frame (Fig. A1). For positive displacement θ corresponding to a specific time (the θ direction is along the z axis), the torque due to gravity at that instant is given by

$$\tau = L\hat{\mathbf{r}} \times m\mathbf{g} = I\ddot{\theta}\hat{\mathbf{z}}, \tag{A1}$$

where the caret (hat) over a variable indicates that it is a unit vector. After evaluating the vector cross product, equation (A1) reduces to the following expression in terms of magnitudes:

$$mL^2\ddot{\theta} + mgL\sin\theta = 0. \tag{A2}$$

The equation of motion of a simple pendulum is in general nonlinear because of the sine term in equation (A2). Unlike archtypical chaotic pendulum motion (Peters, 2007), in seismology applications $\theta_0 \ll 1$ rad is nearly always an acceptable approximation. It should be noted that the pendulum of Figure A1 cannot exhibit chaos because the string supports only tension; this simplification disallows winding modes ($\theta > \pi$) that are part of chaotic motion.

We have assumed that the restoring force is due to a uniform Earth gravitational field of magnitude $g \approx 9.8$ m/sec^2, and that Coriolis acceleration due to the rotating Earth is negligible. If energy is supplied to a simple pendulum to offset its damping, and if the pendulum is located somewhere other than the equator, then the plane of its oscillation is steadily altered because of the Earth's rotation. Aczel (2004) provides a fascinating account of Leon Foucault's invention of this famous pendulum. For pendulums of a type common to seismology, physical constraints against axis rotation make the Coriolis influence inconsequential for most purposes. For a pendulum whose effective length is an appreciable fraction of the Earth radius, such as the Schuler pendulum (Aki and Richards, 2002), the uniform-field approximation is not valid.

Although not shown in Figure A1, further assume that the only other external force acting on the pendulum is one of frictional retardation, viscous damping. With all these approximations, the equation of motion for the nondriven simple pendulum takes on the simple linear form

$$\ddot{\theta} + \frac{\omega_o}{Q}\dot{\theta} + \omega_o^2\theta = 0, \qquad \omega_o^2 = \frac{g}{L}. \tag{A3}$$

This simple-harmonic-oscillator-with-viscous-damping equation is the heart of linear-approximate models of the pendulum and many other mechanical oscillators. It is applicable to the pendulum only to the extent that $\sin\theta \approx \theta$ is acceptable and also that its loss of oscillatory energy derives from damping friction that is proportional to the first power of the velocity. With damping included in the equation of motion

$$\tau = -Lc\dot{\theta} - mgL\sin\theta = mL^2\ddot{\theta}, \tag{A4}$$

where $c\dot{\theta}$ is the viscous friction force, assumed proportional to the angular velocity through the coefficient c, acting on the bob of mass m at a distance L from the axis. It should be noted that c is not always constant (Peters, 2004).

The adjective "mathematical" is appropriate to describe a simple pendulum because real instruments are never as simple as the assumptions made concerning its structure. In addition, a damping term proportional to the velocity does not fully describe the behavior of real oscillators.

The first person to introduce viscous damping to the simple harmonic oscillator may have been physicist Hendrik Anton Lorentz (1853–1928). Neither Lorentz nor George Gabriel Stokes (1819–1903) treated the viscous model as loosely as has been common in recent years (Peters, 2005a).

The quality factor is defined by $Q = -2\pi E/\Delta E$, where E is the energy of oscillation and ΔE is the energy lost per cycle due to the damping. One can easily estimate Q to a few percent from an exponential free decay as follows. After initializing the motion at a given amplitude, count the number of oscillations required for the amplitude to decay to $1/e \approx 0.368$ of the initial value. Q is then obtained by multiplying this number by π.

For linear viscous damping, a simple relationship exists between Q and the damping (decay) coefficient β, used with the exponential to describe the turning points of the motion through $\exp(-\beta t)$:

$$Q = \frac{\omega_o}{2\beta} = \omega_o\frac{mL}{c}. \tag{A5}$$

It should be noted that if β were a constant, then Q would be proportional to the natural frequency f_0 through $\omega_0 = 2\pi f_0$. As demonstrated by Streckeisen, circa 1960, (E. Wielandt, personal comm., 2000) with a LaCoste-spring vertical seismometer, the Q of practical instruments is not proportional to f_0. At least for instruments configured to operate with a long-natural period, the proportionality is one involving f_0^2. For these systems, the damping derives from internal friction in spring and pivot materials, and the best simple model is nonlinear (Peters, 2005a). Although it involves the velocity only by way of algebraic sign, this form of damping is nonlinear, even so resulting in exponential free decay. Although the coefficient β may be reasonably called a damping coefficient, it is not proper to call it a damping constant, because it is not constant but varies with frequency.

Swinging in a fluid such as air, a real pendulum experiences two drag forces, one acting on the bob and the other acting on the string. This problem was first treated analytically by Stokes (1850), originator of the drag-force law $f = 6\pi\eta Rv$ for a small sphere of radius R falling in a liquid at terminal velocity v in a fluid of viscosity η. However, this equation does not in general allow an accurate theoretical estimate of Q based simply on the fluid's viscosity. Stokes' law can be used only when working with very small particles. In particular, for a macroscopic pendulum,the Reynolds number is generally too large to allow its use. In most cases the air influence must include a quadratic velocity term as well as the linear term assumed for equation (A3) (Nelson and Olsson, 1986). In other words, even air damping is not necessarily linear.

In the case of extended rigid bodies undergoing periodic flexure during oscillation, several damping mechanisms generally are present. Internal friction in pivot and structure usually contributes significantly, sometimes dominantly (Peters, 2004). The net quality factor describing the decay is given by

$$\frac{1}{Q_{\text{Net}}} = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots, \tag{A6}$$

where it is seen that the damping mechanism with the lowest quality factor dominates. Only for those mechanisms that give rise to exponential decay is Q independent of the pendulum's amplitude. With quadratic-in-velocity fluid damping, the amplitude trend of Q is opposite to that of Coulomb friction. Unlike hysteretic and viscous damping, neither of these nonlinear mechanisms yields an exponential free decay.

The subscript zero used with ω in equation (A3) is a natural consequence of the mathematical solution to the differential equation. A subscript corresponding to the eigenfrequency of the pendulum in the absence of damping ($Q \to \infty$) is used to distinguish this value from the redshifted frequency when there is damping; that is, $\omega = (\omega_0^2 - \beta^2)^{1/2}$. This redshift is negligible except, perhaps where the pendulum damping is near critical ($Q \approx 0.5$). When internal-friction hysteretic damping is the dominant source of damping, the redshift has no meaning because there is no mechanism to cause it (Peters, 2005a).

Fortunately, when seismic instruments are operated with near-critical damping ($Q = 1/\sqrt{2}$), the assumption of linear viscous damping is for many purposes adequate. Such is the case, for example, when eddy-current damping is employed. It is also true for damping provided by force-balance feedback. In general, however, the damping of an instrument is not governed by a linear equation of motion.

The Driven Simple Pendulum

When the axis of the pendulum undergoes acceleration $a(t)$, application of Newton's laws shows that the zero on the right-hand side of equation (A3) simply is replaced by $-\omega_0^2 a(t)/g$. Conceptual understanding of this term is straightforward. Given that the effect of acceleration **a** and gravitational field **g** are indistinguishable to the pendulum (it measures acceleration at one location only) and that at equilibrium in the absence of **a** the string is oriented along the direction of the acceleration vector **g** from the Earth's gravity, then when there is a constant acceleration **a** of the case perpendicular to **g** the string now aligns itself with the vector sum $\mathbf{g} - \mathbf{a}$. If **g** could be set to zero, alignment would be with$-\mathbf{a}$. Thus, the magnitude of the pendulum deflection angle (for $a \ll g$) is given by $\theta = a/g$. The same result applies for a harmonic acceleration whose frequency is significantly less than the natural frequency of the pendulum. For such an excitation, the derivative terms in the left-hand side of the equation of motion are negligible. In turn it is recognized that $-a/g$ must be multiplied by ω_0^2 to obtain the right-hand side of the now inhomogeneous equation.

Using complex exponentials to describe the steady-state response to harmonic excitation (a pendulum entrained with the drive after transients have settled) yields

$$\left|\frac{g\theta_0}{a_{0,\mathrm{ground}}}\right| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega_0^2\omega^2/Q}}, \qquad \text{(A7)}$$

where $a_{0,\mathrm{ground}}$ is the amplitude of the pivot's acceleration $a(t)$ and is responsible for the angular displacement amplitude of the pendulum θ_0. We refer to equation (A7) as the magnitude of the acceleration transfer function of the simple pendulum, expressed in terms of angular displacement. Other transfer function forms are also considered in this article. Such functions are always a complex steady-state dimensionless ratio. Although axis (pivot) acceleration is the state variable responsible for exciting the pendulum, it is possible to define transfer functions in terms of the other state variables.

I next examine the ratio of the displacement of the pendulum bob to the displacement of the accelerating ground. Because this ratio is complex, it has both a magnitude and a phase. Alternatively, this transfer function could be described in terms of its real and imaginary components though this is rarely done in seismology.

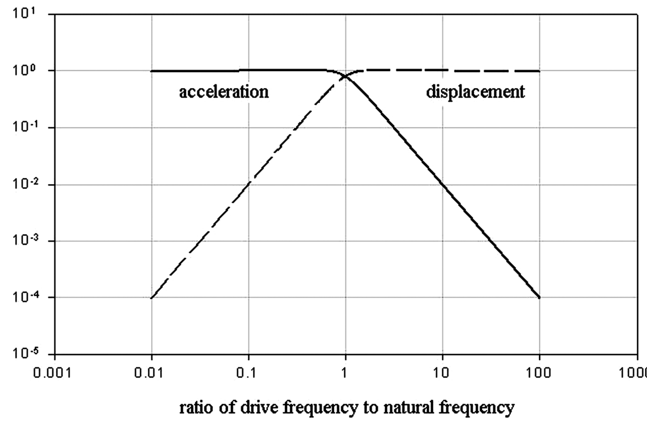

Figure A2. Simple pendulum acceleration response and displacement response as a function of drive frequency ($Q = 1/\sqrt{2}$).

For frequencies above ω_0 it is frequently convenient to work with the displacement transfer function, with magnitude

$$\left|\frac{A_{\mathrm{pendulum}}}{A_{\mathrm{ground}}}\right| = \frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega_0^2\omega^2/Q}}, \qquad \text{(A8)}$$

where for the simple pendulum A_{pendulum} is the amplitude of the bob's displacement, and A_{ground} is the amplitude of sinusoidal ground motion of frequency ω. Figure A2 gives plots of both these transfer functions for quality factor $Q = 1/\sqrt{2}$; Figure A3 shows the phase variation.

Appendix B

Tilt and Rotation

Pendulum Tilt Due to Seismic Surface Waves

The early history of seismometry involved a controversy about tilt (Dewey and Byerly, 1969). Using a simple calculation along with equation (5) in the body of this article, I show

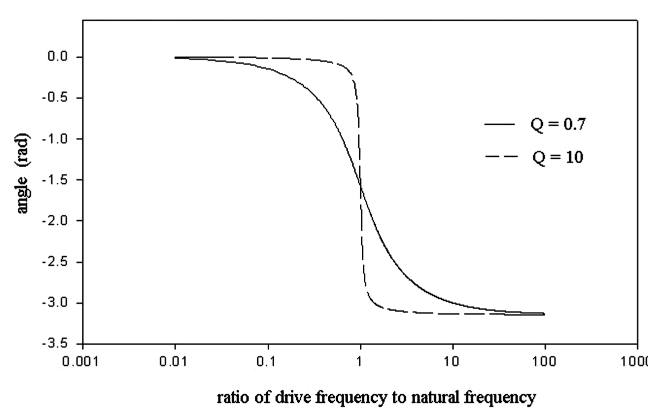

Figure A3. Lag angle of the pendulum, relative to the drive (the case of the instrument), as a function of drive frequency; two different Q-values are shown.

that tilt influence on a pendulum (due, for example, to a surface seismic wave) is negligible compared to the influence of the wave's acceleration components unless the frequency of the wave is very low. For a surface Rayleigh wave having amplitude A, such that

$$y(x, t) = A \sin(kx - \omega t), \qquad k = \frac{2\pi}{\lambda}, \qquad \omega = \frac{2\pi}{T}, \tag{B1}$$

then the maximum of the spatial gradient in a linear elastic solid is given by

$$\left.\frac{\partial y}{\partial x}\right|_{\max} = kA = \frac{2\pi A}{\lambda} = |\theta|_{\text{tilt}}, \tag{B2}$$

the amplitude of the pendulum response to the harmonic tilting that occurs when the wave passes. For ω less than the pendulum's natural frequency ω_o, one finds from equation (5) in the body of this article that the acceleration response of the pendulum is

$$|\theta_{\text{accel}}| = \frac{\omega^2 A}{g}. \tag{B3}$$

Thus, the ratio of tilt-to-acceleration response is

$$\left|\frac{\theta_{\text{tilt}}}{\theta_{\text{accel}}}\right| = \frac{gT}{2\pi v}, \tag{B4}$$

where v is the phase speed of the surface wave. From equation (B4) it is seen that the period of the wave must exceed $2\pi v/g$ for the tilt influence to become greater than the acceleration influence. The crossover period is actually about 50% smaller than the value calculated because the vertical amplitude of particle motion in a Rayleigh wave is roughly 50% greater than the horizontal amplitude. For a wave speed of 2500 m/sec, the crossover period is about 800 sec, beyond which tilt dominates instrument response.

It should be noted that significant deviations in the direction of the Earth's gravitational field occur at frequencies below about 1 mHz as the result of eigenmode oscillations (normal modes). Although the magnitude variations in **g** are exceedingly small, the direction changes in **g** are readily measured with a pendulum acting as a tiltmeter and using a displacement sensor. As will be seen in the discussion of sensor choice, a velocity sensor is not well suited to such measurements. Although the next section is concerned with pendulum measurement of rotation, it should be understood that tilt from Earth normal modes is a special case of rotation in which the direction of the Earth's field changes for an instrument located along a line of nodes. This matter is important because of the need to better understand the mechanisms of Earth hum, which is comprised of such modes. Along with my student M. H. Kwon, around 1990 I accidentally observed persistent oscillation components corresponding to the lowest eigenmode frequencies of the Earth. These observations were made with a pendulum similar to that of Peters (1990), which was designed for surface-physics research. The results were documented by Kwon (Kwon and Peters, 1995).

Pendulum Sensing of Rotation

There is a significant difference in the rotational equation of motion for a pendulum that is restored by a spring, as opposed to a pendulum that is restored by the Earth's gravitational field. Although spring-restored instruments could be used in some cases for measurements in all three axes needed to completely specify the Earth's motion, there is a significant advantage at low frequencies to using gravity-restored instruments for the horizontal axes; measuring rotation around the local vertical axis requires a spring-restored pendulum. The equations developed by Graizer (2006) are restricted in applicability to a spring-restored instrument.

Here we consider a gravity-restored pendulum having an effective length L_{eff}. The homogeneous part (left-hand side) of the equation of motion is identical to equation (5) in the body of this article. Excitation of the pendulum relative to an inertial coordinate frame can only arise from work done by the damping force. In the absence of damping, the pendulum would remain stationary in the inertial frame, while the case holding the instrument oscillated around a horizontal axis. Responding to relative motion between the pendulum and case, the output from the sensor would be the negative of the case motion θ in that inertial frame. With damping, there is motion of both the case and pendulum, and their difference ϕ is what is measured by the sensor. This motion is governed by

$$\ddot{\phi} + \frac{\omega_o}{Q}\dot{\phi} + \omega_o^2\phi = -\ddot{\alpha} - \omega_o^2\alpha, \qquad \omega_o^2 = \frac{g}{L_{\text{eff}}}, \tag{B5}$$

where the Earth rotation variable α is the time varying orientation of the case relative to a horizontal axis in the inertial frame. For a pendulum that is spring restored, the rightmost term of the right-hand side of equation (B5) is missing. It should also be noted (Graizer, 2006), that triaxial instruments of this type generally show cross coupling between orthogonal axes. As before, we can readily predict from equation (B5) the response for both the low- and high-frequency limits. Unlike equation (5) in the body of this article, these limiting cases prove to be identical:

$$\phi = -\alpha, \qquad \text{for } \omega \ll \omega_o \quad \text{and also for} \quad \omega \gg \omega_o. \tag{B6}$$

The full transfer function, obtained as before using the method of Steinmetz phasors, is given by

$$\left|\frac{\phi_o}{\alpha_o}\right| = \frac{\omega_o^2 - \omega^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega_o^2\omega^2/Q^2}}, \qquad \text{phase} = -\tan^{-1}\frac{\omega_o\omega}{Q(\omega_o^2 - \omega^2)}. \tag{B7}$$

The place where the transfer function differs from $\phi = -\alpha$ is in the vicinity of the pendulum's resonance frequency, where a decline in sensor output occurs. For high Q, the near-resonance steady-state decline of the response is narrow. As Q decreases, the width of the region of declination gets larger, as illustrated in Figure B1.

Response, Rotation Compared to Acceleration

The pendulum of Figure 1 in the body of this article can be made sensitive to rotation and simultaneously insensitive to acceleration. As seen from the upper right-hand plot of Figure 3 in the body of this article, moving the axis close to the center of mass causes the pendulum's effective length to be significantly greater than the actual length of the pendulum. In the process, it is seen from equation (14) in the body of this article that the displacement response derived from linear acceleration drops. Because the rotational response is independent of the pendulum's effective length, its sensitivity remains unchanged while the acceleration response is dramatically reduced.

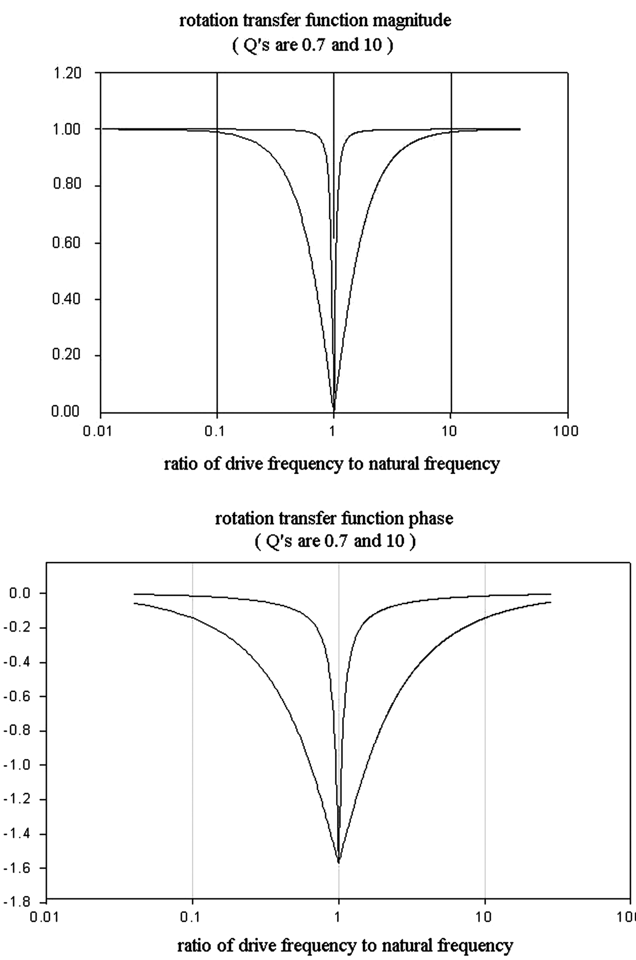

Figure B1. Rotational pendulum transfer function magnitude (upper graph) and phase angle in radians (lower graph).

Choice of Sensor

A truly broadband sensor for measuring all Earth motion is a practical impossibility. Force-balance seismographs, such as those built by Gunar Streckeisen, come as close to this ideal as any. Because these instruments employ velocity sensing, there is a commensurate loss in very low-frequency sensitivity, as can be understood from Figure B2. As can be seen in that figure, if all other things are equal a velocity sensor will outperform a position sensor when detecting Earth motions with characteristic frequencies higher than the natural frequency of the seismic instrument. A position sensor is superior for detecting Earth motions with frequencies lower than the natural frequency of the instrument.

Appendix C

Some Pendulum Characteristics

Kater Pendulum

The rod-only pendulum is useful for recognizing a historically significant property important to precision measurements of g, Earth gravitational acceleration, according to the method of Kater (Peters, 1999). When restricted to measurements involving a single axis of rotation, the accuracy with which the effective length of a pendulum can be measured is limited. This limitation is removed by using a pendulum whose motion is measured for two different, conjugate axes. The pendulum oscillates with the same period when a given first axis is replaced by a second parallel axis passing through the center of percussion calculated from the position of the first axis. The system is equivalent to a simple pendulum of length equal to the distance between the two axes, which can be measured very accurately. The method was used (taking an average of six Kater pendulums) to measure the absolute reference for the Earth's field in Potsdam, Germany, in 1906 (National Geodetic Survey, 1984).

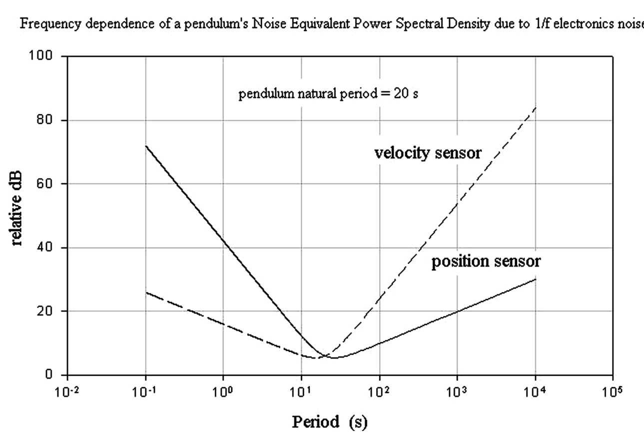

Figure B2. Instrument-self-noise PSD plots of importance to the choice of a sensor. The curves were generated using equation (12) in the body of this article.

Advantages of a Pendulum with $M_1 \neq 0$

Compared to a pendulum of the type shown in Figure 1 in the body of this article, where much of the total mass is concentrated in the dense upper component, M_1, the mechanical integrity of a rod-only pendulum is significantly lower. To obtain the long periods needed for rotational sensing, the axis must be positioned close to the center of the rod. The portion above the axis that results, almost $L/2$ in length, is subject to considerable creep deformation (Peters, 2005b).

Of course the other disadvantage of the rod-only configuration involves the greater size of the instrument. Reducing the height by means of a concentrated upper mass allows a smaller size and reduced cost for the instrument's case. Additionally, the smaller case reduces air-current disturbances due to convective circulation.

Frequency Dependence of Sensitivity

The response of a pendulum is quite different when the drive acceleration at the axis is at an angular frequency significantly lower than or higher than the eigenfrequency. The extremes of these two cases are readily treated by a visual inspection of the equation of motion. From equation (5) in the body of this article we can readily deduce the following:

Low-Frequency Sensitivity Limit

When the drive frequency is very low, the first and second time derivatives of θ are insignificant compared to the remaining term. Thus,

$$\theta = -\frac{a_{\text{axis}}}{g}, \qquad \omega \ll \omega_o. \tag{C1}$$

It is important to understand from equation (C1) that the pendulum's angular sensitivity to acceleration at frequencies well below its natural frequency is independent of the position of the axis. equation (C1) is a general result, no matter the pendulum type. Typically, however, we do not employ a sensor that measures θ directly but rather a detector is placed at the bottom end of the pendulum where it measures the transverse displacement relative to the instrument's case. The amount of motion at the bottom scales with length L, which scales (for a simple pendulum) with the square of the instrument's natural period. Thus, we see that sensitivity to acceleration is proportional to the square of this natural period. To maximize the low-frequency sensitivity of an open-loop (i.e., nonfeedback) system, one should operate with as long a natural period as is conveniently possible, thus the very long natural periods of modern broadband velocity seismometers.

High-Frequency Sensitivity Limit

When the drive frequency is very high, the second derivative term in equation (5) in the body of the article is significantly greater than the other terms on the left-hand side of the equation. Additionally (once transients have decayed away) the pendulum is entrained with the drive and differs in phase by 180°. Entrainment means that the only frequency of pendulum motion is the frequency of the drive. Prior to entrainment, during the transient, both the natural frequency of the pendulum and the frequency of the drive are simultaneously present. Thus, its steady-state frequency is the same as that of $a(t)$, namely ω. Assuming monochromatic simple harmonic motion, we obtain

$$\theta_o = -\frac{\omega_o^2}{g} A_{\text{ground}}, \tag{C2}$$

where A_{ground} is the displacement amplitude (in meters) of the axis and θ_0 is the angular displacement amplitude (in radians) of the pendulum. To put equation (C2) into a more useful form, note that the sensitivity of the instrument depends on where we place the displacement detector. The output of the detector is maximized when its sensing element is placed as far as possible from the axis, that is, at $L - d$ below the axis, the bottom end of the pendulum. We designate the amplitude of the motion there by A_{pendulum} and obtain

$$\frac{A_{\text{pendulum}}}{A_{\text{ground}}} = -\frac{L-d}{L_{\text{eff}}}. \tag{C3}$$

Understanding can be improved by considering equation (C3) for the rod-only pendulum in the particular case $d = 0$. With the axis therefore at one end of the rod and the displacement sensor at its other end, equation (C3) becomes $A_{\text{pendulum}} = -3A_{\text{ground}}/2$. Compared to a simple pendulum of the same length L as the rod, we find that with the sensor at the bottom the compound rod-only pendulum is 50% more sensitive to high-frequency ground displacement. This result follows from considering the center of percussion. At high frequencies ($\omega > \omega_0$), the center of percussion at $2L/3$ is a stationary point in the inertial frame, around which oscillation occurs. In other words, for high frequencies, the center of percussion is also the inertial center of oscillation. A simple drawing consistent with this arrangement reveals the basis for the amplification factor of $3/2$.

Frequency Dependence of the Center of Oscillation

We define the center of oscillation as follows: Consider the line that rotates with the pendulum and that connects the axis and the center of mass as extending to infinity. Further consider the turning point pair of lines in inertial space that are produced when this line is first at $\theta = \theta_0$ and then half a period later at $\theta = -\theta_0$. At high-drive frequencies this pair of lines intersects in inertial space at the center of percussion. As the drive frequency is reduced to below the natural frequency of the pendulum, the intersection of this line-pair moves downward.

The term percussion implies a short lived impulse; as the driving period lengthens, the percussion point is no longer a meaningful reference for the inertial center of rotational motion. The center of oscillation remains meaningful but is no longer located at the center of percussion. As the drive frequency goes toward zero, the center of oscillation moves toward infinity. A (false) assumption that some part of the inertial mass of a seismometer remains stationary in space as the instrument case moves is acceptable when the instrument is functioning as a vibrometer (i.e., drive frequencies above the natural frequency), but it is not true for the low-frequency extreme of the pendulum's response.

Appendix D

Unconventional and Exotic Pendulums

Unconventional Pendulums

Rotation Sensor

Two very different, unconventional gravitational compound pendulums are described in this appendix. Illustrated in Figure D1 is a rotation sensor capable of operating over a broad frequency range. Whereas the pendulum of Figure 1 in the body of this article is a rigid vertical-at-equilibrium structure that oscillates about a horizontal axis, the rigid beam of the pendulum illustrated in Figure D1 is horizontal-at-equilibrium. I believe there are three advantages to this system, although they have not all been experimentally verified. First, the influence of creep is expected to be less for the horizontal configuration as compared to the vertical one. Creep in the members of the long-period vertical pendulum alters the equilibrium position, whereas creep of the boom in the horizontal pendulum alters the period. Secular change in the equilibrium position decreases the maximum possible sensitivity of an instrument's detector, unless force feedback is employed. Period change is inconsequential except as it increases responsiveness to translational acceleration. Because the instrument is designed to minimize this response, the creep influence is of secondary rather than primary importance as in the case of the vertical pendulum.

The second advantage involves air currents. A thermal gradient within the container that holds the instrument can

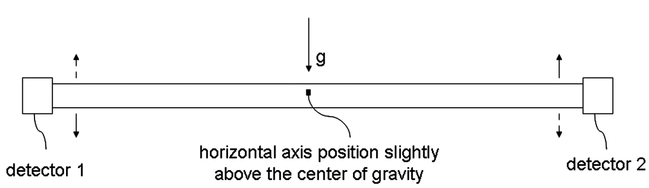

Figure D1. A horizontally oriented gravitational pendulum that is sensitive to rotation but insensitive to translation. The axis of rotation is perpendicular to the plane of the figure, and the arrows show directions of the end-arm motion.

cause convective flows, and the resulting circulation is expected to have greater influence on the vertical pendulum than on the horizontal pendulum.

The final advantage results from the simple means by which one mounts a pair of displacement detectors on opposite ends of the beam. Operating differentially in phase opposition, they yield a better signal-to-noise ratio (SNR) than is possible from a single detector. The greater the length of the beam, the greater will be the sensitivity of the instrument.

Some Other Inertial Rotation Sensors

Parts from a pair of STS-1 horizontal seismometers were used to build a rotation sensor with mechanical properties similar to the rotation pendulums I have described. The instrument tested by Hutt *et al.* (2004) differs, however, by containing springs; the lower flexures are placed under tension by means of a large brass counterweight.

Morrissey (2000) also built a beam-balance broadband tiltmeter with similar mechanical properties. His instrument used a pair of lead masses mounted on opposite ends of a horizontal aluminum bar suspended at the center with a flexural axis. It used force-feedback balancing and he claimed a sensitivity of 120 $\mathrm{mV}/\mu\mathrm{rad}$, with a resolution of better than 0.1 nrad, using linear variable differential transformers (LVDT). Wielandt (2002) notes that a capacitive sensor is superior to an LVDT for the reason of the granular nature of ferromagnetism of the latter. Compared to a capacitive sensor of (singly) differential type which is customary, there is an SNR advantage to using a pair of (doubly = fully) differential capacitive sensors with the pendulum of Figure D1, one such fully-differential detector for each end, with the pair operating in phase opposition.

Microseism Detector

Although conventional seismographs always operate with damping near 0.7 or 0.8, an undamped vertical pendulum with merit is next described. Valuable information concerning hurricanes (via microseisms) could be gleaned from a large network of inexpensive pendulums operating with a reasonably high Q. Lengthening the period by moving the center of mass close to the axis has the following advantage. The sensitivity of the pendulum to frequencies other than resonance is significantly decreased as shown in Figure D2. This is especially important for high-frequency noises that derive from localized, cultural disturbances. This would allow the SNR of the electronics employed to be relaxed without a significant loss of microseism detectability (Fig. D2).

The transient response of this high-Q pendulum would disallow meaningful analysis of time domain data; however, analyses in the frequency domain, using power spectral density (PSD) plots or cumulative spectral power (CSP) plots would not be similarly limited (Peters, 2008a,b). Knowledge of the Q, used to correct the spectra in generating the PSD, allows for the generation of a wealth of useful information.

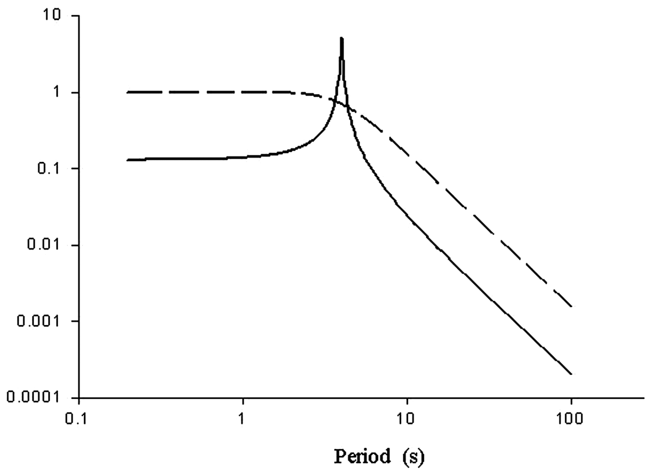

Figure D2. The displacement response of a compound pendulum tuned to resonate with a Q of 40 at a period of 4 s. The response of a near-critically damped simple pendulum of comparable physical length is shown for comparison (dashed curve).

Exotic Pendulums

Appreciation for the physics of pendulum dynamics is improved by consideration of unconventional instruments such as the Schuler pendulum and a gravity-gradient-stabilized satellite that behaves like a pendulum

Schuler Pendulum

Tuned to a period of 84 min, the Schuler pendulum is important to navigation. That period matches near-earth satellite periods and is described in detail elsewhere (Aki and Richards, 2002).

Gravity-Gradient-Stabilized-Satellite Pendulum

Because of the inverse-square dependence of the Earth's field, a cigar-shaped satellite behaves as a very long-period pendulum. The radial gradient of the field is responsible for a separation of the center of mass and the center of gravity, so that a torque exists when the long axis deviates from the vertical. The method selected by De Moraes and Da Silva (1990) to treat this problem is based on Hamiltonian dynamics. The following analysis uses Newton's second law and his expression for the gravitational field.

Assume a body in the shape of the rod-only pendulum discussed in Appendix C, having mass m and length L. The center of mass is located at the center of the rod, and the center of gravity is calculated using

$$\int_0^L \frac{GM_e dm}{(R+y)^2} = \frac{mGM_e}{(R+y_g)^2}, \qquad dm = \frac{m}{L}dy, \qquad \text{(D1)}$$

where $G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$ is the Newtonian constant, $M_e = 6 \times 10^{24}$ kg is the mass of the Earth, $R = 6.4 \times 10^6$ m is the radius of the Earth, y is measured from the bottom of the rod, and y_g is the center of gravity. The integral of equation (D1) yields to first order the expression

$$y_g \approx \frac{L}{2} - \frac{L^2}{2R}. \qquad \text{(D2)}$$

Application of Newton's second law yields

$$I_c\ddot{\theta} + mg\frac{L^2}{2R}\theta \approx 0, \qquad I_c = \frac{mL^2}{12}, \qquad \text{(D3)}$$

from which it is seen that $\omega_0^2 = 6g/R$. With $g \approx 9.2\ \mathrm{m/sec^2}$ at an altitude of 200 km, the pendulum's effective length is $R/6$, yielding a period of about 36 min. Oscillation in the absence of imposed damping would prevent the satellite from accomplishing mission objectives. A method that has been used to attenuate oscillation employs hysteretic damping derived from electrical currents that are powered by solar cells.

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