Continuous probability distributions

- Many continuous probability distributions, including:
  - Uniform
  - Normal
  - Gamma
  - Exponential
  - Chi-Squared
  - Lognormal
  - Weibull

Uniform distribution

- Simplest – characterized by the interval endpoints, $A$ and $B$.

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \leq x \leq B \\ 0 & \text{elsewhere} \end{cases}$$

- Mean and variance:

$$\mu = \frac{A + B}{2} \quad \text{and} \quad \sigma^2 = \frac{(B - A)^2}{12}$$
Example

A circuit board failure causes a shutdown of a computing system until a new board is delivered. The delivery time $X$ is uniformly distributed between 1 and 5 days.

What is the probability that it will take 2 or more days for the circuit board to be delivered?

Normal distribution

- The “bell-shaped curve”
- Also called the Gaussian distribution
- The most widely used distribution in statistical analysis
  - forms the basis for most of the parametric tests we’ll perform later in this course.
  - describes or approximates most phenomena in nature, industry, or research
- Random variables ($X$) following this distribution are called normal random variables.
  - the parameters of the normal distribution are $\mu$ and $\sigma$ (sometimes $\mu$ and $\sigma^2$.)
Normal distribution

- The density function of the normal random variable \( X \), with mean \( \mu \) and variance \( \sigma^2 \), is

\[
n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{all } x.
\]

![Normal Distribution](image)

Standard Normal RV ...

- Note: the probability of \( X \) taking on any value between \( x_1 \) and \( x_2 \) is given by:

\[
P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)dx = \int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx
\]

- To ease calculations, we define a normal random variable

\[
Z = \frac{X - \mu}{\sigma}
\]

where \( Z \) is normally distributed with \( \mu = 0 \) and \( \sigma^2 = 1 \)
Standard Normal Distribution

- Table A.3: “Areas Under the Normal Curve”

Examples

- $P(Z \leq 1) =$

- $P(Z \geq -1) =$

- $P(-0.45 \leq Z \leq 0.36) =$
Your turn ...

- Use Table A.3 to determine (draw the picture!)
  1. \( P(Z \leq 0.8) = \)
  
  2. \( P(Z \geq 1.96) = \)
  
  3. \( P(-0.25 \leq Z \leq 0.15) = \)
  
  4. \( P(Z \leq -2.0 \text{ or } Z \geq 2.0) = \)

The normal distribution “in reverse”

- Example:
  Given a normal distribution with \( \mu = 40 \) and \( \sigma = 6 \), find the value of \( X \) for which 45% of the area under the normal curve is to the left of \( X \).

  1) If \( P(Z < k) = 0.45 \),

  \( k = \) _________

  2) \( Z = \) _______

  \( X = \) _______
Normal approximation to the binomial

- If $n$ is large and $p$ is not close to 0 or 1, or
  - if $n$ is smaller but $p$ is close to 0.5, then

  the binomial distribution can be approximated by the normal distribution using the transformation:

  $$Z = \frac{(X \pm 0.5) - np}{\sqrt{npq}}$$

- NOTE: add or subtract 0.5 from $X$ to be sure the value of interest is included (draw a picture to know which)
- Look at example 6.15, pg. 191


Look at example 6.15, pg. 191

$p = 0.4 \quad n = 100$

$\mu = \quad \sigma = \quad$

if $x = 30$, then $z = \quad$

and, $P(X < 30) = P (Z < \quad) = \quad$
Your turn

- Refer to the previous example,

1. What is the probability that more than 50 survive?

2. What is the probability that exactly 45 survive?

Gamma & exponential distributions

- Recall the Poisson Process
  - Number of occurrences in a given interval or region
  - “Memoryless” process
- Sometimes we’re interested in the time or area until a certain number of events occur.
- For example
  - An average of 2.7 service calls per minute are received at a particular maintenance center. The calls correspond to a Poisson process.
    - What is the probability that up to a minute will elapse before 2 calls arrive?
    - How long before the next call?
Gamma Distribution

- The density function of the random variable $X$ with gamma distribution having parameters $\alpha$ (number of occurrences) and $\beta$ (time or region).

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha - 1} e^{-\frac{x}{\beta}}, \quad x > 0.$$  

$$\Gamma(n) = (n-1)!$$

$$\mu = \alpha \beta$$

$$\sigma^2 = \alpha \beta^2$$

Exponential distribution

- Special case of the gamma distribution with $\alpha = 1$.

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0.$$  

✓ Describes the time until or time between Poisson events.

$$\mu = \beta$$

$$\sigma^2 = \beta^2$$
Example

An average of 2.7 service calls per minute are received at a particular maintenance center. The calls correspond to a Poisson process.

What is the probability that up to a minute will elapse before 2 calls arrive?

\[ \beta = \quad \alpha = \quad \]

\[ P(X \leq 1) = \quad \]

Example (cont.)

What is the expected time before the next call arrives?

\[ \beta = \quad \alpha = \quad \]

\[ \mu = \quad \]
Chi-squared distribution

• Special case of the gamma distribution with \( \alpha = \nu/2 \) and \( \beta = 2 \).

\[
f(x) = \frac{1}{2^{\nu/2} \Gamma(\nu/2)} \frac{x^{\nu-1} e^{-x/2}}{2^{\nu/2} \Gamma(\nu/2)} \quad x > 0.
\]

where \( \nu \) is a positive integer.

✓ single parameter; \( \nu \) is called the degrees of freedom.

\[
\mu = \nu
\]

\[
\sigma^2 = 2\nu
\]
Lognormal distribution

- When the random variable $Y = \ln(X)$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, then $X$ has a lognormal distribution with the density function,

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma^2 x}} e^{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}}, \quad x \geq 0$$

$$\mu = e^{\mu + \sigma^2/2}$$

$$\sigma^2 = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

Example

Look at problem 6.72, pg. 207 ...

Since $\ln(X)$ has normal distribution with $\mu = 5$ and $\sigma = 2$, the probability that $X > 50,000$ is,

$$P(X > 50,000) = \text{______________________________}$$
Wiebull distribution

- Used for many of the same applications as the gamma and exponential distributions, but
  - does not require memoryless property of the exponential

\[ f(x; \alpha, \beta) = \alpha \beta x^{\beta-1} e^{-\alpha x^\beta}, \quad x > 0 \]

\[ F(x) = 1 - e^{-\alpha x^\beta} \]

Example

- Designers of wind turbines for power generation are interested in accurately describing variations in wind speed, which in a certain location can be described using the Weibull distribution with \( \alpha = 0.02 \) and \( \beta = 2 \). A designer is interested in determining the probability that the wind speed in that location is between 3 and 7 mph.

\[ P(3 < X < 7) = \]