Two-Dimension Problems: Stress Formulation

Unknown field quantities: functions of *x* and *y*

Plane Strain

$\sigma_x, \sigma_y, \tau_{xy}$

$\varepsilon_x, \varepsilon_y, \gamma_{xy}$

σ_z

u, v

Plane Stress

$\sigma_x,\ \sigma_y,\ \tau_{xy}$

$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy}$

ε_z

$u,\ v$

Governing Equations

Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

Compatibility

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

Boundary Conditions

$$p_x = \sigma_x \ell + \tau_{xy} m$$

$$p_y = \tau_{xy} \ell + \sigma_y m$$

Airy Stress Function Formulation (***F*** = 0)

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \quad \Rightarrow \quad \text{Automatically satisfy equilibrium}$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Compatibility yields the single governing equation: $\nabla^4 \Phi = 0$

plus the boundary conditions on stress enable solution for the three in-plane stress components

Strains are obtained from isotropic Hooke's Law

$$\varepsilon_x = \frac{1+\nu}{E}\left[(1-\nu)\sigma_x - \nu\sigma_y\right]$$

$$\varepsilon_y = \frac{1+\nu}{E}\left[(1-\nu)\sigma_y - \nu\sigma_x\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

Displacements are obtained from strain-displacement relations:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Two-Dimensional Problems in Polar Coordinates: *(r,ϑ)*

Equilibrium Equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} + F_\theta = 0$$

Strain-Displacement Equations

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$$

Compatibility equations in terms of stress:

Plane Strain

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{1}{1-\nu}\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right)$$

Plane Stress

$$\nabla^2(\sigma_r + \sigma_\theta) = -(1+\nu)\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right)$$

where: $$\nabla^2(\ ) = \frac{\partial^2(\ )}{\partial r^2} + \frac{1}{r}\frac{\partial(\ )}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\ )}{\partial \theta^2}$$

Airy Stress Function (zero body forces)

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \qquad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\nabla^4 \Phi = 0$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

Boundary conditions, Hooke's Law, transformation equations have same form as expressions in Cartesian coordinates. replacing *x* by *r*, *y* by *ϑ*

Axisymmetric Problems

General Solution for Airy Stress Function: $\Phi = K_1 r^2 \ln r + K_2 r^2 + K_3 \ln r + K_4$

Stresses

$$\sigma_r = \frac{1}{r}\frac{d\Phi}{dr} = K_1\left(2\ln r + 1\right) + 2K_2 + K_3\frac{1}{r^2} \qquad \sigma_\theta = \frac{d^2\Phi}{dr^2} = K_1(2\ln r + 3) + 2K_2 - K_3\frac{1}{r^2} \qquad \tau_{r\theta} = 0$$

Strains (plane stress)

$$\varepsilon_r = \frac{1}{E}\left[K_1\left(2\left(1-\nu\right)\ln r + 1 - 3\nu\right) + 2K_2\left(1-\nu\right) + K_3\left(1+\nu\right)\frac{1}{r^2}\right]$$

$$\varepsilon_\theta = \frac{1}{E}\left[K_1\left(2\left(1-\nu\right)\ln r + 3 - \nu\right) + 2K_2\left(1-\nu\right) - K_3\left(1+\nu\right)\frac{1}{r^2}\right]$$

$$\gamma_{r\theta} = 0$$

Displacements (plane stress)

$$u = \frac{1}{E}\left[K_1\left(2\left(1-\nu\right)r\ln r - \left(1+\nu\right)r\right) + 2K_2\left(1-\nu\right)r - K_3\frac{\left(1+\nu\right)}{r}\right] + A\sin\theta + B\cos\theta$$

$$v = \frac{4}{E}K_1 r\theta + A\cos\theta - B\sin\theta + Cr$$