Kinematics of Displacement and Deformation: Strain

A body is deformed/displaced from its initial configuration to its final configuration (see Figure 1). A point p in the initial configuration has coordinates x_i; the same point moves to P in the final configuration, with coordinates X_i. A nearby point to p, denoted as q, is a distance ds from p in the direction of the unit vector $\boldsymbol{n}$. Point q moves to Q in the final configuration; it is a distance dS from P in the direction of the unit vector $\boldsymbol{N}$.

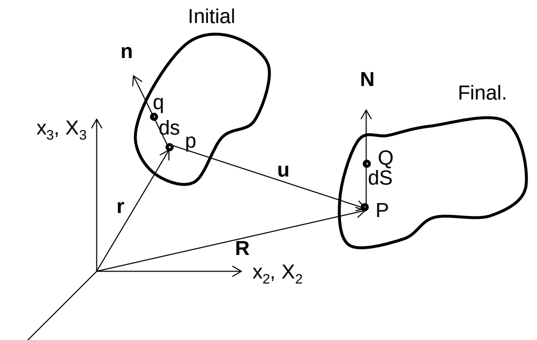

Figure 1.

$$\boldsymbol{r} = x_i \boldsymbol{e}_i \tag{1}$$

$$d\boldsymbol{r} = dx_i \boldsymbol{e}_i = ds\boldsymbol{n} = dsn_i \boldsymbol{e}_i \tag{2}$$

$$\boldsymbol{R} = X_i \boldsymbol{e}_i \tag{3}$$

$$d\boldsymbol{R} = dX_i \boldsymbol{e}_i = dS\boldsymbol{N} = dSN_i \boldsymbol{e}_i \tag{4}$$

The general change in configuration from initial (undeformed) to final (deformed) may be thought of as a one-to-one mapping of points from initial position x_i to final position X_i; i.e.,

$$X_i = X_i(x_1, x_2, x_3) \tag{5}$$

or the inverse,

$$x_i = x_i(X_1, X_2, X) \tag{6}$$

From which we may write the differentials,

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j \tag{7}$$

$$dx_i = \frac{\partial x_i}{\partial X_j} dX_j \tag{8}$$

If we choose the choose the x_i (undeformed coordinates) as the independent variables, and express the X_i in terms of them, as in (5), we are employing the so-called Lagrangian, or material, description of the deformation.

If we choose the choose the X_i (deformed coordinates) as the independent variables, and express the x_i in terms of them, as in (6), we are employing the so-called Eulerian, or spatial, description of the deformation.

I. <u>Extensional strain</u> – Change in length of a line element

The square of the length of a line element before and after deformation is:

From (2) and (8): $(ds)^2 = |d\boldsymbol{r}|^2 = dx_k dx_k = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} dX_i dX_j$

From (4) and (7): $(dS)^2 = |d\boldsymbol{R}|^2 = dX_k dX_k = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j$

The change in the square of the length is:

$$\begin{aligned}(dS)^2 - (ds)^2 &= \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j - dx_i dx_i \\ &= \left(\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} - \delta_{ij}\right) dx_i dx_j \qquad \text{(Lagrangian)} \qquad (9)\end{aligned}$$

$$(dS)^2 - (ds)^2 = \left(\delta_{ij} - \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\right) dX_i dX_j \qquad \text{(Eulerian)} \qquad (10)$$

Define the Lagrangian strain tensor

$$\boxed{\varepsilon_{ij}^L = \frac{1}{2}\left(\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} - \delta_{ij}\right)} \qquad (11)$$

Then, by (9)

$$\frac{(dS)^2 - (ds)^2}{(ds)^2} = 2\varepsilon_{ij}^L \frac{dx_i}{ds}\frac{dx_j}{ds} = 2\varepsilon_{ij}^L n_i n_j$$

Therefore, let

$$\varepsilon_{(n)}^L = \frac{1}{2}\frac{(dS)^2 - (ds)^2}{(ds)^2} = \varepsilon_{ij}^L n_i n_j \qquad (12)$$

be defined as the Lagrangian measure of extensional strain of a line element in the direction $\boldsymbol{n}$ in the undeformed configuration.

Similarly, define the Eulerian strain tensor

$$\boxed{\varepsilon_{ij}^E = \frac{1}{2}\left(\delta_{ij} - \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\right)} \qquad (13)$$

Then, by (10)

$$\frac{(dS)^2 - (ds)^2}{(dS)^2} = 2\varepsilon_{ij}^E \frac{dX_i}{dS}\frac{dX_j}{dS} = 2\varepsilon_{ij}^E N_i N_j$$

Therefore, let

$$\varepsilon_{(n)}^{E} = \frac{1}{2}\frac{(dS)^2-(ds)^2}{(dS)^2} = \varepsilon_{ij}^{E} N_i N_j \tag{14}$$

be defined as the Eulerian measure of extensional strain of a line element in the direction of ***N*** in the deformed configuration.

By their definitions, both the Lagrangian strain tensor (11) and the Eulerian strain tensor (13) are symmetric, second order tensors, and therefore transform under a coordinate rotation according to the rules of tensor transformation.

II. <u>Components of strain in terms of displacements</u>

The displacement vector is defined (Figure 1) as:

$$\boldsymbol{u} = \boldsymbol{R}-\boldsymbol{r}$$

or, in terms of components,

$$u_i = X_i - x_i$$

If the displacement components are expressed in terms of the undeformed coordinates as independent variables, then the following derivatives m y be defined:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial X_i}{\partial x_j} - \delta_{ij} \text{ ; or} \qquad \frac{\partial X_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \tag{15}$$

Alternatively, if the displacement components are expressed in terms of the deformed coordinates as independent variables, then the following derivatives may be defined:

$$\frac{\partial u_i}{\partial X_j} = \delta_{ij} - \frac{\partial x_i}{\partial X_j} \text{; or} \qquad \frac{\partial x_i}{\partial X_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} \tag{16}$$

Substitute (15) into (11):

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial u_k}{\partial x_i}\right)\left(\delta_{kj} + \frac{\partial u_k}{\partial x_j}\right) - \delta_{ij}\right]$$

$$= \frac{1}{2}\left[\delta_{ki}\delta_{kj} + \delta_{ki}\frac{\partial u_k}{\partial x_j} + \frac{\partial u_k}{\partial x_i}\delta_{kj} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \delta_{ij}\right]$$

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \tag{17}$$

Similarly, by substituting (16) into (13), we get

$$\varepsilon_{ij}^E = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} - \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right] \tag{18}$$

Therefore, components of each of our strains tensors may be expressed in terms of displacement gradients (derivatives of displacement components with respect to position).

From now on, we will confine our discussion to the Lagrangian strain tensor ε_{ij}^L (we could perform a parallel development on the Eulerian strain tensor, and obtain similar results). We may thus employ the notation $(\,)_{,i} = \frac{\partial()}{\partial x_i}$ without ambiguity; that is all derivatives are with respect to the undeformed coordinates x_1, x_2, x_3, and all field quantities are expressed in terms of the undeformed coordinates. Therefore, equation (14) may be written as

$$\varepsilon_{ij}^L = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{19}$$

III. Shear strain: changes in angle between line elements.

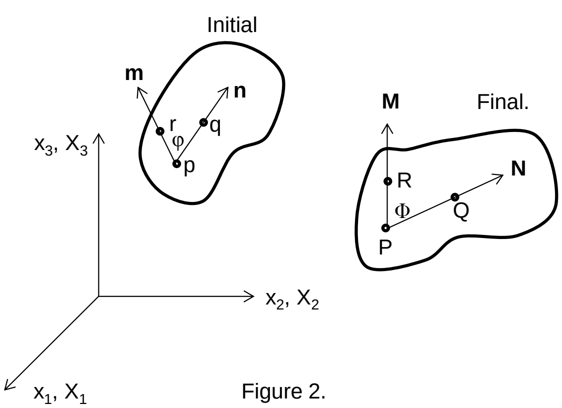

Figure 2.

$\boldsymbol{n} \bullet \boldsymbol{m} = \cos\varphi$

$\boldsymbol{N} \bullet \boldsymbol{M} = \cos\Phi$

$N_i = \frac{dX_i}{dS}$ (from (4))

$n_i = \frac{dx_i}{ds}$ (from (2))

$$N_i = \frac{dX_i}{dS} = \frac{dX_i}{ds}\frac{ds}{dS} = \left(\frac{\partial X_i}{\partial x_j}\frac{dx_j}{ds}\right)\frac{ds}{dS} \quad \text{so that} \quad \frac{dS}{ds}N_i = \frac{\partial X_i}{\partial x_j}\frac{dx_j}{ds}$$

Recall,

$$\varepsilon_{(n)}^L = \frac{1}{2}\frac{(dS)^2 - (ds)^2}{(ds)^2} = \frac{1}{2}\left(\left(\frac{dS}{ds}\right)^2 - 1\right) \quad \text{so that} \quad \frac{dS}{ds} = (1 + 2\varepsilon_n)^{1/2}$$

Substitute into the above to get:

$$\left(1 + 2\varepsilon_{(n)}^L\right)^{1/2} N_i = \frac{\partial X_i}{\partial x_j} n_j \tag{20}$$

Similarly, $$\left(1+2\varepsilon^L_{(m)}\right)^{1/2} M_i = \frac{\partial X_i}{\partial x_j} m_j \tag{21}$$

Multiply (20) by (21)

$$\left(1+2\varepsilon^L_{(n)}\right)^{1/2}\left(1+2\varepsilon^L_{(m)}\right)^{1/2} N_i M_i = \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k} n_j m_k$$

$$\left(1+2\varepsilon^L_{(n)}\right)^{1/2}\left(1+2\varepsilon^L_{(m)}\right)^{1/2} N_i M_i = \left(\frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k} - \delta_{jk} + \delta_{jk}\right) n_j m_k$$

$$\left(1+2\varepsilon^L_{(n)}\right)^{1/2}\left(1+2\varepsilon^L_{(m)}\right)^{1/2} cos\Phi = 2\varepsilon^L_{jk} n_j m_k + cos\varphi$$

Therefore, let

$$\varepsilon^L_{(nm)} \equiv \tfrac{1}{2}\left(1+2\varepsilon^L_{(n)}\right)^{1/2}\left(1+2\varepsilon^L_{(m)}\right)^{1/2} cos\Phi - \tfrac{1}{2}\ cos\varphi = \varepsilon^L_{ij} n_i m_j \tag{22}$$

be defined as the Lagrangian measure of shear strain between two line elements in the directions ***n*** and ***m*** in the undeformed configuration.

IV. Components of strain for a body undergoing a general rigid displacement.

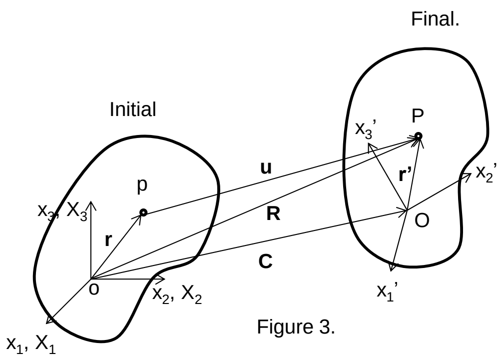

Figure 3.

x_1', x_2', x_3' axes fixed in the body

$\boldsymbol{C} = C_i\boldsymbol{e_i}$ translation of o

$\boldsymbol{r} = x_i\boldsymbol{e_i}$; $\boldsymbol{R} = X_i\boldsymbol{e_i} = \boldsymbol{C} + \boldsymbol{r'}$

$\boldsymbol{r'} = x_j'\boldsymbol{e_j}` = x_j\boldsymbol{e_j}` = x_j\alpha_{ji}\boldsymbol{e_i}$

where α_{ji} is the rotation matrix defining the rotation of the body.

Therefore,

$$\boldsymbol{R} = (C_i + x_j\alpha_{ji})\boldsymbol{e_i}$$

The displacement of point p is

$$\mathbf{u} = \mathbf{R} - \mathbf{r}$$

$$u_i = C_i + x_j\alpha_{ji} - x_i = C_i + (\alpha_{ji} - \delta_{ji})x_j$$

This is the displacement of an arbitrary point in the body, expressed in terms of its coordinates in the initial configuration (Lagrangian description).

The displacement gradients are

$$u_{i,k} = (\propto_{ji} - \delta_{ji})x_{j,k} = (\propto_{ji} - \delta_{ji})\delta_{jk} = \propto_{ki} - \delta_{ki}$$

We will substitute this into equation (19), the Lagrangian strain tensor in terms of the displacements:

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

which gives

$$\varepsilon_{ij}^{L} = \frac{1}{2}\Big(\propto_{ji} - \delta_{ji} + \propto_{ij} - \delta_{ij} + (\propto_{ik} - \delta_{ik})\left(\propto_{jk} - \delta_{jk}\right)\Big)$$

$$= \frac{1}{2}\left(\propto_{ji} - \delta_{ji} + \propto_{ij} - \delta_{ij} + \propto_{ik}\propto_{jk} - \propto_{ik}\delta_{jk} - \delta_{ik}\propto_{jk} + \delta_{ik}\delta_{jk}\right)$$

$$= \frac{1}{2}\left(\propto_{ji} - \delta_{ji} + \propto_{ij} - \delta_{ij} + \delta_{ij} - \propto_{ij} - \propto_{ji} + \delta_{ij}\right)$$

$$\varepsilon_{ij}^{L} = 0$$

Therefore, the Lagrangian strain tensor corresponding to a general rigid displacement is zero.

The differential of displacement may be written as:

$$du_i = u_{i,j}dx_j = \left[\frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) + \right]dx_j$$

$$du_i = \left[\varepsilon_{ij}^{L} + \omega_{ij}^{L}\right]dx_j$$

where

$$\omega_{ij}^{L} \equiv \frac{1}{2}\left(u_{i,j} - u_{j,i} - u_{k,i}u_{k,j}\right)$$

is the Lagrangian rotation tensor. As can be seen from the foregoing, for a rigid-body motion

$$\omega_{ij}^{L} = \alpha_{ji} - \delta_{ji} \qquad \text{and} \qquad u_i = C_i + \left(\alpha_{ji} - \delta_{ji}\right)x_j = C_i + \omega_{ij}^{L}x_j$$

V. Physical interpretation of the components of the Lagrangian strain tensor

A. Diagonal components. Recall that, from equation (12), the Lagrangian extensional strain, defined as one-half the change in the square of the length divided by the square of the original length of a line element initially in the direction ***n*** is

$$\varepsilon_{(n)}^{L} = \frac{1}{2}\frac{(dS)^2 - (ds)^2}{(ds)^2} = \varepsilon_{ij}^{L} n_i n_j$$

If we let ***n*** = ***e₁***, then

$$\varepsilon_{(1)}^{L} = \varepsilon_{11}^{L}$$

In other words the 1-1 component of the Lagrangian strain tensor represents the Lagrangian extensional strain of a line element initially in the x_1 direction. Similarly, in the x_2 and x_3 directions:

$$\varepsilon_{(2)}^{L} = \varepsilon_{22}^{L}$$
$$\varepsilon_{(3)}^{L} = \varepsilon_{33}^{L}$$

B. Off-diagonal components. Recall that, from equation (22), the Lagrangian shear strain between two line elements initially in the directions of unit vectors ***n*** and ***m*** is:

$$\varepsilon_{(nm)}^{L} \equiv \frac{1}{2}\left(1 + 2\varepsilon_{(n)}^{L}\right)^{1/2}\left(1 + 2\varepsilon_{(m)}^{L}\right)^{1/2} cos\Phi - \frac{1}{2}\, cos\varphi = \varepsilon_{ij}^{L} n_i m_j$$

If we let ***n*** = $\boldsymbol{e_1}$ and ***m*** = $\boldsymbol{e_2}$ then

$$\varepsilon_{(nm)}^{L} \equiv \frac{1}{2}(1 + 2\varepsilon_{11}^{L})^{1/2}(1 + 2\varepsilon_{22}^{L})^{1/2} cos\Phi = \varepsilon_{12}^{L}$$

Similarly in the 2-3 and 1-3 directions., So the off-diagonal terms of the Lagrangian strain tensor represent the shear strains between two line elements initially oriented in the coordinate directions.

VI. Small displacement gradients.

The formulation to this point gives an exact description of the geometry of displacements and deformations; no restrictions have been placed on the magnitudes of the deformation nor on the rigid-body motion. Consider now the case in which displacement gradients are small; that is

$$\frac{\partial u_i}{\partial x_j} \ll 1 \qquad \text{and} \qquad \frac{\partial u_i}{\partial X_j} \ll 1$$

This means that the difference in displacement between two neighboring points is small in comparison to the distance between the points. If this is the case, we can make the following observations.

A. In the Lagrangian strain tensor:

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$$

the product terms are much smaller than the first two terms on the left hand side, and may be neglected, leading to the so-called "linear strain tensor"

$$\varepsilon_{ij}^{L} \cong \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

This may be considered an approximation to the "exact" Lagrangian strain tensor. We term it this linear strain tensor because its components are linear functions of the displacement components.

B. Consider some arbitrary function F expressed in terms of the deformed coordinates, $F(X_1,X_2,X_3)$. If we calculate its derivatives with respect to the undeformed coordinates x_i, we get

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial X_j}\frac{\partial X_j}{\partial x_i} = \frac{\partial F}{\partial X_j}\left(\frac{\partial u_j}{\partial x_i} + \delta_{ij}\right) = \frac{\partial F}{\partial X_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial F}{\partial X_i}$$

If displacement gradients are small, then the second term on the right side is much larger than the first, and

$$\frac{\partial F}{\partial x_i} \cong \frac{\partial F}{\partial X_i}$$

That is, in the case of small displacement gradients, it is immaterial whether a derivative is calculated with respect to the deformed coordinates (X_i) or the undeformed coordinates (x_i). In this case, the Eulerian strain tensor:

$$\varepsilon_{ij}^{E} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} - \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right] \cong \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right] \cong \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

$$\varepsilon_{ij}^{E} \cong \varepsilon_{ij}^{L} \cong \varepsilon_{ij} \cong \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

So, the Eulerian strain tensor is essentially indistinguishable from the Lagrangian strain tensor; both are approximated by the linear strain tensor; and the two descriptions of the deformation (material and spatial) coincide.

C. Our basic definitions of strain.

1) Extensional strain.

$$\varepsilon_{(n)} \equiv \frac{1}{2}\frac{(dS)^2 - (ds)^2}{(ds)^2} = \varepsilon_{ij}n_i n_j \ll 1$$

$$\varepsilon_{(n)} = \frac{1}{2}\frac{(dS - ds)(dS + ds)}{(ds)^2} \cong \frac{1}{2}\frac{(dS - ds)(2ds)}{(ds)^2} = \frac{(dS - ds)}{ds}$$

That is, the extensional strain may be interpreted as change in length divided by original length, which is the usual engineering definition of extensional strain. The diagonal terms on the linear strain tensor therefore represent the relative elongation of line elements in the coordinate directions.

2) Shear strain

$$\varepsilon_{(nm)} \equiv \frac{1}{2}\left(1 + 2\varepsilon_{(n)}\right)^{1/2}\left(1 + 2\varepsilon_{(m)}\right)^{1/2} cos\Phi - \frac{1}{2}\ cos\varphi = \varepsilon_{ij}n_i m_j \ll 1$$

$$\varepsilon_{(nm)} \cong \frac{1}{2}(1)(1)cos\Phi - \frac{1}{2}\ cos\varphi = \frac{1}{2}(cos\Phi - cos\varphi)$$

Consider the case when ***n*** is perpendicular to ***m***; then $\varphi = \pi/2$ and cos$\varphi = 0$.
Let $\Delta\varphi = \varphi - \Phi = \pi/2 - \Phi$; then

$$\varepsilon_{(nm)} \cong \frac{1}{2}\cos\left(\frac{\pi}{2} - \Delta\varphi\right) = \frac{1}{2}\sin\Delta\varphi \cong \frac{1}{2}\Delta\varphi$$

So that $\quad \varepsilon_{(nm)} \cong \frac{1}{2}\Delta\varphi \quad$ for ***n*** $\perp$ ***m***

That is, when ***n*** is perpendicular to ***m***, $\varepsilon_{(nm)}$ is ½ the change in angle between these initially perpendicular lines. The off-diagonal components of the strain tensor therefore represent ½ the change in angle between line elements in coordinate directions.

D. Rotations

For small displacement gradients, the rotation tensor can be approximated:

$$\omega_{ij}^{L} = \frac{1}{2}\left(u_{i,j} - u_{j,i} - u_{k,i}u_{k,j}\right) \cong \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \ll 1$$

The infinitesimal rotation vector is therefore defined as

$$\omega_{ij} \equiv \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

Note that this is an anti-symmetric tensor: diagonal components are zero, and there are only three independent components.

Therefore, we can define an associated vector, **ω**, such that

$$\omega_{ij} = -\varepsilon_{ijk}\omega_k$$

This equation can be inverted by multiplying both sides by ε_{ijl} :

$$\omega_{ij}\varepsilon_{ijl=} - \varepsilon_{ijk}\varepsilon_{ijl}\omega_k = -2\delta_{kl}\omega_k = -2\omega_l$$

so that

$$\omega_l = -\frac{1}{2}\varepsilon_{ijl}\omega_{ij}$$

or, specifically,

$$\omega_1 = \omega_{32} = \frac{1}{2}\left(u_{3,2} - u_{2,3}\right)$$
$$\omega_2 = \omega_{13} = \frac{1}{2}\left(u_{1,3} - u_{3,1}\right)$$
$$\omega_3 = \omega_{21} = \frac{1}{2}\left(u_{2,1} - u_{1,2}\right)$$

ω is referred to as the <u>infinitesimal rotation vector</u>.

Recall, for a general rigid displacement,

$$u_i = C_i + \left(\propto_{ji} - \delta_{ji}\right)x_j = C_i + \omega_{ij}^L x_j$$

Now, for an infinitesimal rotation,

$$\omega_{ij}^L \cong \omega_{ij} = -\varepsilon_{ijk}\omega_k$$

so that

$$u_i = C_i - \varepsilon_{ijk}x_j\omega_k = C_i + \varepsilon_{ijk}\omega_k x_j$$

Or, in general vector notation $\boldsymbol{u} = \boldsymbol{C} + \boldsymbol{\omega}\times\boldsymbol{r}$

E. Based on the above discussions, "small displacement gradients" has the physical interpretation of small deformations (strain) <u>and</u> small rotations. Only when <u>both</u> strains and rotations are small are we justified in using the linearized kinematic relations.