MAE 691 Continuum Mechanics

H.W. #10 Solution

1. Uniaxial, incompressible

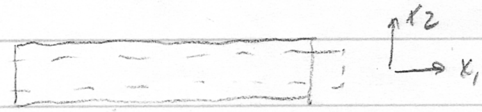
Mooney - Rivlin

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3)$$

$$x_1 = \alpha X_1$$

$$x_2 = \frac{1}{\sqrt{\alpha}} X_2$$

$$x_3 = \frac{1}{\sqrt{\alpha}} X_3$$



$$F = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\alpha}} \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

$$I_1 = B_{11} + B_{22} + B_{33} = \boxed{\alpha^2 + \frac{2}{\alpha}}$$

$$I_2 = \frac{1}{2}(B_{ii}B_{jj} - B_{ij}B_{ij}) = B_{11}B_{22} + B_{22}B_{33} + B_{11}B_{33} - B_{12}^2 - B_{23}^2 - B_{23}^2$$

$$I_2 = \alpha + \alpha + \frac{1}{\alpha^2} = \boxed{\frac{1}{\alpha^2} + 2\alpha}$$

Constitutive:

$$T_{ij} = -p\,\delta_{ij} + 2W_1 B_{ij} - 2W_2 B_{ij}^{-1}$$

$$T_{11} = -p + 2C_{10}\alpha^2 - 2C_{01}\frac{1}{\alpha^2}$$

$$T_{22} = T_{33} = -p + 2C_{10}\frac{1}{\alpha} - 2C_{01}\alpha$$

For stress-free lateral surfaces:

$$T_{22} = T_{33} = 0 = -p + 2C_{10}\frac{1}{\alpha} - 2C_{01}\alpha \qquad p = 2C_{10}\frac{1}{\alpha} - 2C_{01}\alpha$$

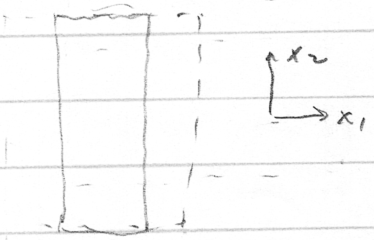
$$T_{11} = -2C_{10}\frac{1}{\alpha} + 2C_{01}\alpha + 2C_{10}\alpha^2 - 2C_{01}\frac{1}{\alpha^2}$$

$$= 2C_{10}\left(\alpha^2 - \frac{1}{\alpha}\right) + 2C_{01}\left(\alpha - \frac{1}{\alpha^2}\right)$$

$$\boxed{T_{11} = 2\left(C_{10} + \frac{1}{\alpha}C_{01}\right)\left(\alpha^2 - \frac{1}{\alpha}\right)}$$

2. *Strip biaxial*

$$x_1 = \alpha X_1$$
$$x_2 = X_2$$
$$x_3 = \frac{1}{\alpha} X_3$$



$$F = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\alpha^2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

$$\boxed{I_1 = \alpha^2 + 1 + \frac{1}{\alpha^2}}$$
$$\boxed{I_2 = 1/\alpha^2 + 1 + \alpha^2}$$

$$T_{ij} = -p\delta_{ij} + 2C_{10} B_{ij} - 2C_{01} B_{ij}^{-1}$$
$$T_{11} = -p + 2C_{10}\alpha^2 - 2C_{01} 1/\alpha^2$$
$$T_{22} = -p + 2C_{10} - 2C_{01}$$
$$T_{33} = -p + 2C_{10}\frac{1}{\alpha^2} - 2C_{01}\alpha^2$$

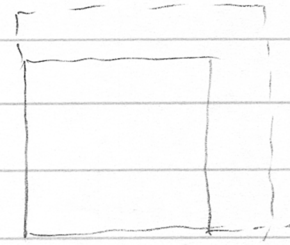
x_3 surface free $\Rightarrow$ $T_{33} = 0 = -p + 2C_{10}\frac{1}{\alpha^2} - 2C_{01}\alpha^2$

$$p = 2C_{10}\frac{1}{\alpha^2} - 2C_{01}\alpha^2$$

$$T_{11} = -2C_{10}\frac{1}{\alpha^2} + 2C_{01}\alpha^2 + 2C_{10}\alpha^2 - 2C_{01}\frac{1}{\alpha^2}$$
$$= 2C_{10}\left(-\frac{1}{\alpha^2} + \alpha^2\right) + 2C_{01}\left(\alpha^2 - \frac{1}{\alpha^2}\right)$$
$$\boxed{T_{11} = 2(C_{10} + C_{01})\left(\alpha^2 - \frac{1}{\alpha^2}\right)}$$

$$T_{22} = -2C_{10}\frac{1}{\alpha^2} + 2C_{01}\alpha^2 + 2C_{10} - 2C_{01}$$
$$= 2C_{10}\left(-\frac{1}{\alpha^2} + 1\right) + 2C_{01}\left(\alpha^2 - 1\right)$$
$$\boxed{T_{22} = 2\left(C_{10} + \alpha^2 C_{01}\right)\left(1 - \frac{1}{\alpha^2}\right)}$$

3. Equibiaxial



$$x_1 = \alpha X_1$$

$$x_2 = \alpha X_2$$

$$x_3 = \frac{1}{\alpha^2} X_3$$

$$F = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1/\alpha^2 \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 1/\alpha^4 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & 1/\alpha^2 & 0 \\ 0 & 0 & \alpha^4 \end{bmatrix}$$

$$\boxed{\begin{aligned} I_1 &= \frac{1}{\alpha^4} + 2\alpha^2 \\ I_2 &= \alpha^4 + \frac{2}{\alpha^2} \end{aligned}}$$

$$T_{ij} = -p\,\delta_{ij} + 2C_{10} B_{ij} - 2C_{01} B_{ij}^{-1}$$

$$T_{11} = -p + 2C_{10}\alpha^2 - 2C_{10}\frac{1}{\alpha^2}$$

$$T_{22} = T_{11}$$

$$T_{33} = -p + 2C_{10}\frac{1}{\alpha^4} - 2C_{01}\alpha^4$$

For stress-free x_3-surface. $\quad 0 = -p + 2C_{10}\frac{1}{\alpha^4} - 2C_{01}\alpha^4$

$$p = 2C_{10}\frac{1}{\alpha^4} - 2C_{01}\alpha^4$$

So $\quad T_{11} = T_{22} = -2C_{10}\frac{1}{\alpha^4} + 2C_{01}\alpha^4 + 2C_{10}\alpha^2 - 2C_{10}\frac{1}{\alpha^2}$

$$= 2C_{10}\left(\alpha^2 - \frac{1}{\alpha^4}\right) + 2C_{01}\left(\alpha^4 - \frac{1}{\alpha^2}\right)$$

$$\boxed{T_{11} = T_{22} = 2\left(C_{10} + \alpha^2 C_{01}\right)\left(\alpha^2 - \frac{1}{\alpha^4}\right)}$$