Pure Bending:
Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane.
Other Loading Types

- **Eccentric Loading**: Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple.

- **Transverse Loading**: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple.

- **Principle of Superposition**: The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.
• Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section bending moment.

• From statics, a couple \( M \) consists of equal and opposite forces.

• The sum of the components of the forces in any direction is zero.

• The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.

• These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

\[
F_x = \int \sigma_x \, dA = 0 \\
M_y = \int z \sigma_x \, dA = 0 \\
M_z = \int -y \sigma_x \, dA = M
\]
Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a neutral surface must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it
Consider a beam segment of length $L$.

After deformation, the length of the neutral surface remains $L$. At other sections,

\[
L' = (\rho - y)\theta
\]

\[
\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta
\]

\[
\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho}
\]  
(strain varies linearly)

$y$ is measured from the neutral surface

$\rho$ is radius of curvature of neutral surface
Stress Due to Bending

- For a linearly elastic material,
  \[ \sigma_x = E \varepsilon_x = -E \frac{y}{\rho} \]
  (stress varies linearly)

- For static equilibrium,
  \[ F_x = 0 = \int \sigma_x \, dA = \int -E \frac{y}{\rho} \, dA \]
  \[ 0 = -\frac{E}{\rho} \int y \, dA \]
  First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.

- For static equilibrium,
  \[ M = \int (-y \sigma_x) \, dA = \int (-y) \left(-E \frac{y}{\rho}\right) \, dA \]
  \[ M = \frac{E}{\rho} \int y^2 \, dA = \frac{E I_z}{\rho} \]
  \[ \frac{1}{\rho} = \frac{M}{E I_z} \]
  Substituting into \( \sigma_x = -E \frac{y}{\rho} \)
  \[ \sigma_x = -\frac{My}{I_z} \]
Symmetry of Cross-Section

- For static equilibrium,

\[ 0 = \int (z \sigma_x \, dA) = \int (z) \left( -\frac{M_y}{I_z} \right) dA \]

\[ 0 = \frac{M}{I_z} \int y_z \, dA \]

- The integral is zero if the cross-section is symmetric about either the y- or z-axis

- The flexure formula:

\[ \sigma_x = -\frac{M_y}{I_z} \]

is limited to:

- Linear elastic material
- Bending about an axis of symmetry of the cross-section, or about an axis perpendicular to an axis of symmetry
 Beam Section Properties

• The maximum normal stress due to bending,

\[ \sigma_m = \frac{Mc}{I} = \frac{M}{S} \]

\( I = \text{section moment of inertia} \)

\( S = \frac{I}{c} = \text{section modulus} \)

A beam section with a larger section modulus will have a lower maximum stress.

• Consider a rectangular beam cross section,

\[ S = \frac{I}{c} = \frac{1}{12}bh^3 = \frac{1}{6}bh^3 = \frac{1}{6}Ah \]

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

• Structural steel beams are designed to have a large section modulus.
### Appendix C. Properties of Rolled-Steel Shapes
(SI Units)

**S Shapes**
(American Standard Shapes)

| Designation | Area, \( A, \text{mm}^2 \) | Depth, \( d, \text{mm} \) | Flange Width, \( b_y, \text{mm} \) | Flange Thickness, \( t_b, \text{mm} \) | Web Thickness, \( t_w, \text{mm} \) | Axis X-X \( I_x \times 10^6 \text{mm}^4 \) | Axis X-X \( S_x \times 10^3 \text{mm}^3 \) | Axis X-X \( r_x, \text{mm} \) | Axis Y-Y \( I_y \times 10^6 \text{mm}^4 \) | Axis Y-Y \( S_y \times 10^3 \text{mm}^3 \) | Axis Y-Y \( r_y, \text{mm} \) |
|-------------|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| S610 \( \times \) 180 | 22900 | 622 | 204 | 27.7 | 20.3 | 1320 | 4240 | 240 | 34.9 | 341 | 39.0 |
| 158 | 20100 | 622 | 200 | 27.7 | 15.7 | 1230 | 3950 | 247 | 32.5 | 321 | 39.9 |
| 149 | 19000 | 610 | 184 | 22.1 | 18.9 | 995 | 3260 | 229 | 20.2 | 215 | 32.3 |
| 134 | 17100 | 610 | 181 | 22.1 | 15.9 | 938 | 3080 | 234 | 19.0 | 206 | 33.0 |
| 119 | 15200 | 610 | 178 | 22.1 | 12.7 | 878 | 2880 | 240 | 17.9 | 198 | 34.0 |
| S510 \( \times \) 143 | 18200 | 516 | 183 | 23.4 | 20.3 | 700 | 2710 | 196 | 21.3 | 228 | 33.9 |
| 128 | 16400 | 516 | 179 | 23.4 | 16.8 | 658 | 2550 | 200 | 19.7 | 216 | 34.4 |
| 112 | 14200 | 508 | 162 | 20.2 | 16.1 | 530 | 2090 | 193 | 12.6 | 152 | 29.5 |
| 98.3 | 12500 | 508 | 159 | 20.2 | 12.8 | 495 | 1950 | 199 | 11.8 | 145 | 30.4 |
| S460 \( \times \) 104 | 13300 | 457 | 159 | 17.6 | 18.1 | 385 | 1685 | 170 | 10.4 | 127 | 27.5 |
| 81.4 | 10400 | 457 | 152 | 17.6 | 11.7 | 333 | 1460 | 179 | 8.83 | 113 | 28.8 |
| S380 \( \times \) 74 | 9500 | 381 | 143 | 15.6 | 14.0 | 201 | 1060 | 145 | 6.65 | 90.8 | 26.1 |
| 64 | 8150 | 381 | 140 | 15.8 | 10.4 | 185 | 971 | 151 | 6.15 | 85.7 | 27.1 |
Sample Problem 4.2

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

\[
\bar{Y} = \frac{\sum yA}{\sum A} \quad I_x' = \sum (I + Ad^2)
\]

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

\[
\sigma_m = \frac{Mc}{I}
\]

- Calculate the curvature

\[
\frac{1}{\rho} = \frac{M}{EI}
\]

A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing \( E = 165 \) GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.
Sample Problem 4.2

SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

\[
\begin{array}{ccc}
\text{Area, mm}^2 & \bar{y}, \text{mm} & \bar{y}A, \text{mm}^3 \\
1 & 20 \times 90 = 1800 & 50 & 90 \times 10^3 \\
2 & 40 \times 30 = 1200 & 20 & 24 \times 10^3 \\
\hline
\Sigma A = 3000 & \Sigma \bar{y}A = 114 \times 10^3 \\
\end{array}
\]

\[
\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{114 \times 10^3}{3000} = 38 \text{ mm}
\]

\[
I_x = \sum (\bar{I} + A d^2) = \sum \left( \frac{1}{12}bh^3 + A d^2 \right)
\]

\[
= \left( \frac{1}{12} \times 20 \times 90 \times 20^3 + 1800 \times 12^2 \right) + \left( \frac{1}{12} \times 30 \times 40^3 + 1200 \times 18^2 \right)
\]

\[
I = 868 \times 10^3 \text{ mm}^4 = 868 \times 10^{-9} \text{ m}^4
\]
Sample Problem 4.2

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

\[ \sigma_m = \frac{Mc}{I} \]

\[ \sigma_A = \frac{M c_A}{I} = \frac{3 \text{kN} \cdot \text{m} \times 0.022 \text{m}}{868 \times 10^{-9} \text{m}^4} = 76.0 \text{ MPa} \]

\[ \sigma_B = -\frac{M c_B}{I} = -\frac{3 \text{kN} \cdot \text{m} \times 0.038 \text{m}}{868 \times 10^{-9} \text{m}^4} = -131.3 \text{ MPa} \]

- Calculate the curvature

\[ \frac{1}{\rho} = \frac{M}{EI} \]

\[ = \frac{3 \text{kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{m}^4)} \]

\[ \frac{1}{\rho} = 20.95 \times 10^{-3} \text{ m}^{-1} \]

\[ \rho = 47.7 \text{ m} \]
Consider a composite beam formed from two materials with $E_1$ and $E_2$.

Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \quad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

Define a transformed section such that

$$dF_2 = -\frac{(nE_1) y}{\rho} dA = -\frac{E_1 y}{\rho} (n dA) \quad n = \frac{E_2}{E_1}$$
Example 4.03

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass

- Evaluate the cross sectional properties of the transformed section

- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.

- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.

Bar is made from bonded pieces of steel ($E_s = 29 \times 10^6$ psi) and brass ($E_b = 15 \times 10^6$ psi). Determine the maximum stress in the steel and brass when a moment of 40 kip*in is applied.
Example 4.03

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

\[
\begin{align*}
E_s &= \frac{E_s}{E_b} = \frac{29 \times 10^6 \text{ psi}}{15 \times 10^6 \text{ psi}} = 1.933 \\
& b_T = 0.4 \text{ in} + 1.933 \times 0.75 \text{ in} + 0.4 \text{ in} = 2.25 \text{ in}
\end{align*}
\]

- Evaluate the transformed cross sectional properties

\[
I = \frac{1}{12} b_T h^3 = \frac{1}{12} (2.25 \text{ in})(3 \text{ in.})^3 = 5.063 \text{ in.}^4
\]

- Calculate the maximum stresses

\[
\begin{align*}
\sigma_m &= \frac{Mc}{I} = \frac{(40 \text{ kip} \cdot \text{in.})(1.5 \text{ in.})}{5.063 \text{ in.}^4} = 11.85 \text{ ksi} \\
(\sigma_b)_{\text{max}} &= \sigma_m = 11.85 \text{ ksi} \\
(\sigma_s)_{\text{max}} &= n\sigma_m = 1.933 \times 11.85 \text{ ksi} = 22.9 \text{ ksi}
\end{align*}
\]
Stress concentrations may occur:

- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

\[ \sigma_m = K \frac{Mc}{I} \]
Members Made of an Elastoplastic Material

- Rectangular beam made of an elastoplastic material
  \[ \sigma_x \leq \sigma_Y \quad \sigma_m = \frac{Mc}{I} \]
  \[ \sigma_m = \sigma_Y \quad M_Y = \frac{I}{c} \sigma_Y = \text{maximum elastic moment} \]

- If the moment is increased beyond the maximum elastic moment, plastic zones develop around an elastic core.
  \[ M = \frac{3}{2} M_Y \left( 1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right) \quad y_Y = \text{elastic core half-thickness} \]

- In the limit as the moment is increased further, the elastic core thickness goes to zero, corresponding to a fully plastic deformation.
  \[ M_p = \frac{3}{2} M_Y = \text{plastic moment} \]
  \[ k = \frac{M_p}{M_Y} = \text{shape factor (depends only on cross section shape)} \]
Plastic Deformations of Members With a Single Plane of Symmetry

- Fully plastic deformation of a beam with only a vertical plane of symmetry.
- The neutral axis cannot be assumed to pass through the section centroid.
- Resultants $R_1$ and $R_2$ of the elementary compressive and tensile forces form a couple.
  
  \[ R_1 = R_2 \]
  
  \[ A_1 \sigma_Y = A_2 \sigma_Y \]
  
  The neutral axis divides the section into equal areas.
- The plastic moment for the member,
  
  \[ M_p = \left( \frac{1}{2} A \sigma_Y \right) d \]
Residual Stresses

- Plastic zones develop in a member made of an elastoplastic material if the bending moment is large enough.

- Since the linear relation between normal stress and strain applies at all points during the unloading phase, it may be handled by assuming the member to be fully elastic.

- Residual stresses are obtained by applying the principle of superposition to combine the stresses due to loading with a moment $M$ (elastoplastic deformation) and unloading with a moment $-M$ (elastic deformation).

- The final value of stress at a point will not, in general, be zero.
A member of uniform rectangular cross section is subjected to a bending moment $M = 36.8$ kN-m. The member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa.

Determine (a) the thickness of the elastic core, (b) the radius of curvature of the neutral surface.

After the loading has been reduced back to zero, determine (c) the distribution of residual stresses, (d) radius of curvature.
Example 4.05, 4.06

- Thickness of elastic core:

\[ M = \frac{3}{2} M_Y \left( 1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right) \]

\[ 36.8 \text{kN} \cdot \text{m} = \frac{3}{2} (28.8 \text{kN} \cdot \text{m}) \left( 1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right) \]

\[ \frac{y_Y}{c} = \frac{y_Y}{60 \text{mm}} = 0.666 \quad 2y_Y = 80 \text{mm} \]

- Radius of curvature:

\[ \varepsilon_Y = \frac{\sigma_Y}{E} = \frac{240 \times 10^6 \text{Pa}}{200 \times 10^9 \text{Pa}} \]

\[ = 1.2 \times 10^{-3} \]

\[ \varepsilon_Y = \frac{y_Y}{\rho} \]

\[ \rho = \frac{y_Y}{\varepsilon_Y} = \frac{40 \times 10^{-3} \text{m}}{1.2 \times 10^{-3}} \quad \rho = 33.3 \text{m} \]

\[ \frac{I}{c} = \frac{2}{3} bc^2 = \frac{2}{3} (50 \times 10^{-3} \text{m}) (60 \times 10^{-3} \text{m})^2 \]

\[ = 120 \times 10^{-6} \text{m}^3 \]

\[ M_Y = \frac{I}{c} = \frac{1}{\sigma_Y} = \left( 120 \times 10^{-6} \text{m}^3 \right) (240 \text{MPa}) \]

\[ = 28.8 \text{kN} \cdot \text{m} \]
Example 4.05, 4.06

- $M = 36.8$ kN-m
  - $y_Y = 40$ mm
  - $\sigma_Y = 240$ MPa

- $M = -36.8$ kN-m
  - $\sigma'_m = \frac{M c}{I} = \frac{36.8 \text{kN} \cdot \text{m}}{120 \times 10^6 \text{m}^3} = 306.7$ MPa $< 2\sigma_Y$

- $M = 0$
  - At the edge of the elastic core,
    - $\epsilon_x = \frac{\sigma_x}{E} = \frac{-35.5 \times 10^6 \text{Pa}}{200 \times 10^9 \text{Pa}} = -177.5 \times 10^{-6}$
    - $\rho = -\frac{y_Y}{\epsilon_x} = \frac{40 \times 10^{-3} \text{m}}{177.5 \times 10^{-6}} = 225 \text{m}$
Eccentric Axial Loading in a Plane of Symmetry

- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due to a pure bending moment
  \[ \sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}} \]
  \[ = \frac{P}{A} - \frac{My}{I} \]

- Eccentric loading
  \[ F = P \]
  \[ M = Pd \]

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.
Example 4.07

SOLUTION:

- Find the equivalent centric load and bending moment

- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.

- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.

- Find the neutral axis by determining the location where the normal stress is zero.

An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis.
Example 4.07

- Normal stress due to a centric load
  \[ A = \pi c^2 = \pi (0.25\text{in})^2 \]
  \[ = 0.1963\text{in}^2 \]
  \[ \sigma_0 = \frac{P}{A} = \frac{160\text{lb}}{0.1963\text{in}^2} \]
  \[ = 815\text{psi} \]

- Equivalent centric load and bending moment
  \[ P = 160\text{lb} \]
  \[ M = Pd = (160\text{lb})(0.65\text{in}) \]
  \[ = 104\text{lb}\cdot\text{in} \]

- Normal stress due to bending moment
  \[ I = \frac{1}{4}\pi c^4 = \frac{1}{4}\pi (0.25)^4 \]
  \[ = 3.068\times10^{-3}\text{in}^4 \]
  \[ \sigma_m = \frac{Mc}{I} = \frac{(104\text{lb}\cdot\text{in})(0.25\text{in})}{3.068\times10^{-3}\text{in}^4} \]
  \[ = 8475\text{psi} \]
Example 4.07

- Maximum tensile and compressive stresses
  \[ \sigma_t = \sigma_0 + \sigma_m \]
  \[ = 815 + 8475 \]
  \[ \sigma_t = 9260 \text{ psi} \]
  \[ \sigma_c = \sigma_0 - \sigma_m \]
  \[ = 815 - 8475 \]
  \[ \sigma_c = -7660 \text{ psi} \]

- Neutral axis location
  \[ 0 = \frac{P}{A} - \frac{My_0}{I} \]
  \[ y_0 = \frac{P}{A} \frac{I}{M} = (815 \text{ psi}) \frac{3.068 \times 10^{-3} \text{ in}^4}{105 \text{ lb} \cdot \text{in}} \]
  \[ y_0 = 0.0240 \text{ in} \]
Unsymmetric Bending

- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple.
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.
Unsymmetric Bending

Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.
  \[ M_z = M \cos \theta \quad M_y = M \sin \theta \]

- Superpose the component stress distributions
  \[ \sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} \]

- Along the neutral axis,
  \[ \sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} = -\frac{(M \cos \theta)y}{I_z} + \frac{(M \sin \theta)y}{I_y} \]
  \[ \tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta \]
SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.
  \[ M_z = M \cos \theta \quad M_y = M \sin \theta \]

- Combine the stresses from the component stress distributions.
  \[ \sigma_x = \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \]

- Determine the angle of the neutral axis.
  \[ \tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta \]
Example 4.08

- Resolve the couple vector into components and calculate the corresponding maximum stresses.

\[
M_z = (1600 \text{ lb} \cdot \text{in}) \cos 30 = 1386 \text{ lb} \cdot \text{in} \\
M_y = (1600 \text{ lb} \cdot \text{in}) \sin 30 = 800 \text{ lb} \cdot \text{in} \\
I_z = \frac{1}{12} (1.5 \text{ in})(3.5 \text{ in})^3 = 5.359 \text{ in}^4 \\
I_y = \frac{1}{12} (3.5 \text{ in})(1.5 \text{ in})^3 = 0.9844 \text{ in}^4
\]

The largest tensile stress due to \( M_z \) occurs along \( AB \)

\[
\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386 \text{ lb} \cdot \text{in})(1.75 \text{ in})}{5.359 \text{ in}^4} = 452.6 \text{ psi}
\]

The largest tensile stress due to \( M_z \) occurs along \( AD \)

\[
\sigma_2 = \frac{M_y z}{I_y} = \frac{(800 \text{ lb} \cdot \text{in})(0.75 \text{ in})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}
\]

- The largest tensile stress due to the combined loading occurs at \( A \).

\[
\sigma_{\text{max}} = \sigma_1 + \sigma_2 = 452.6 + 609.5 = 1062 \text{ psi}
\]
Example 4.08

- Determine the angle of the neutral axis.

\[
\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30
\]

\[
= 3.143
\]

\[
\phi = 72.4^\circ
\]