Sixth Edition

CHAPTER

3

MECHANICS OF MATERIALS

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Torsion

Lecture Notes:

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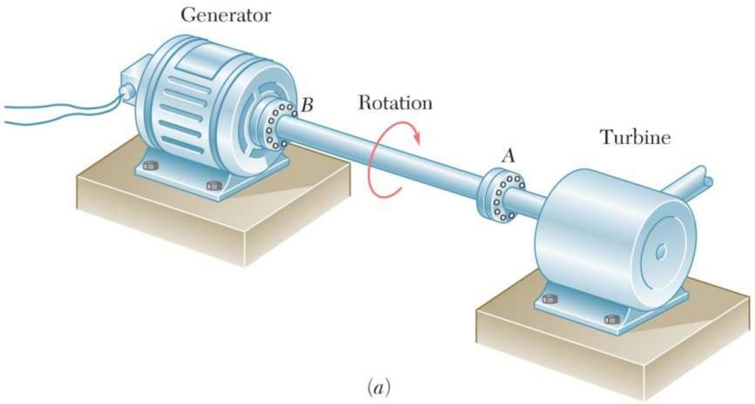
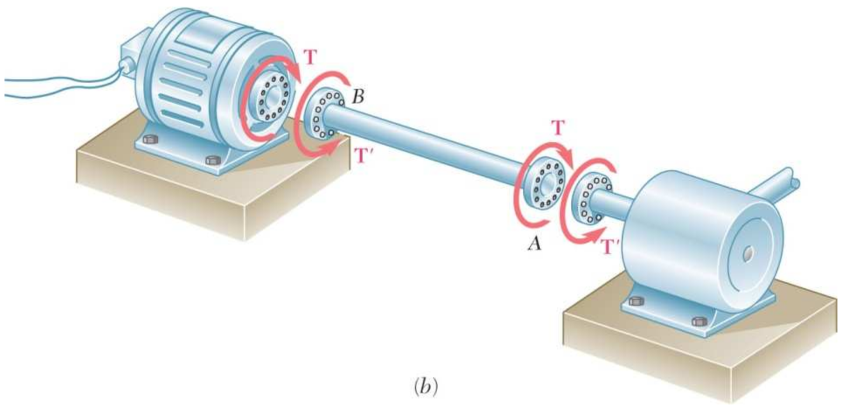
Torsional Loads on Circular Shafts



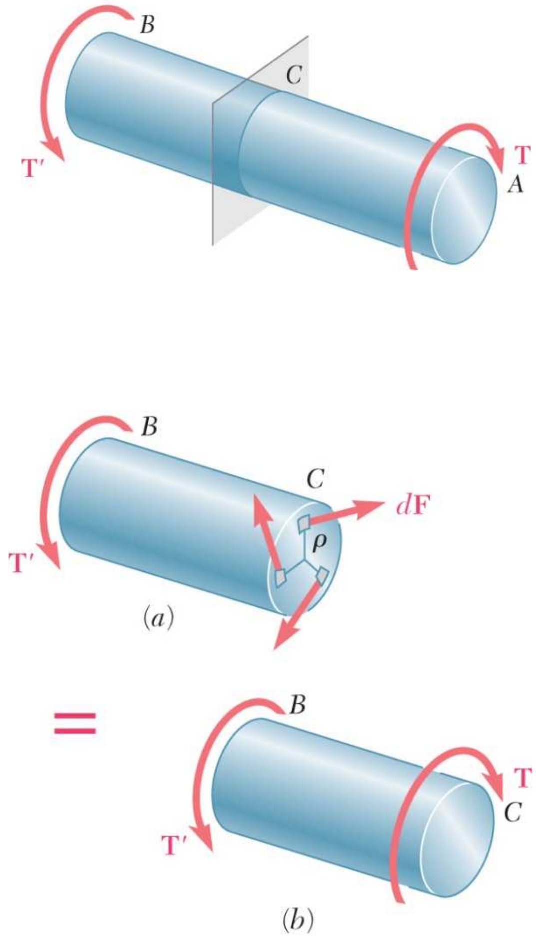
(a)



(b)

- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*'

Net Torque Due to Internal Stresses



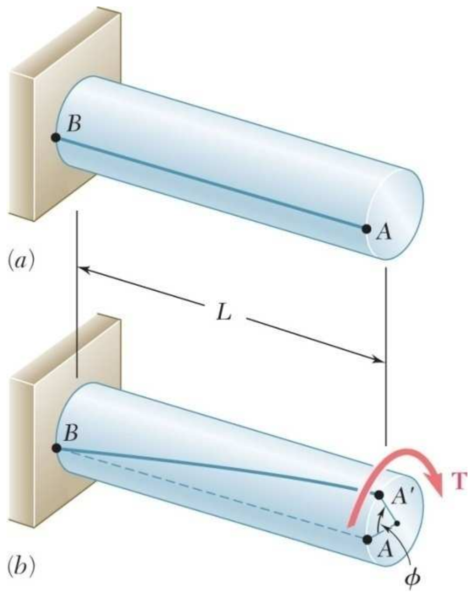
- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

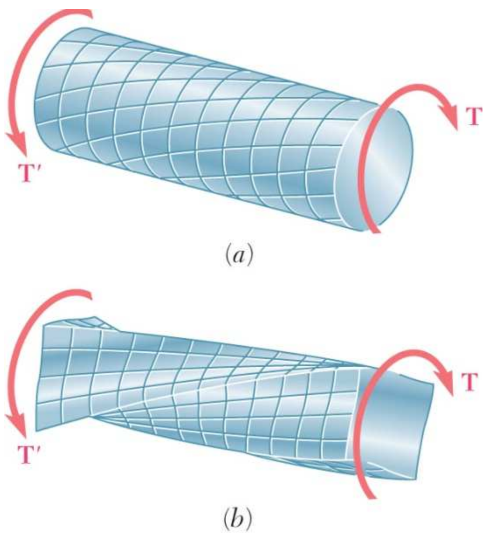
End

Shaft Deformations



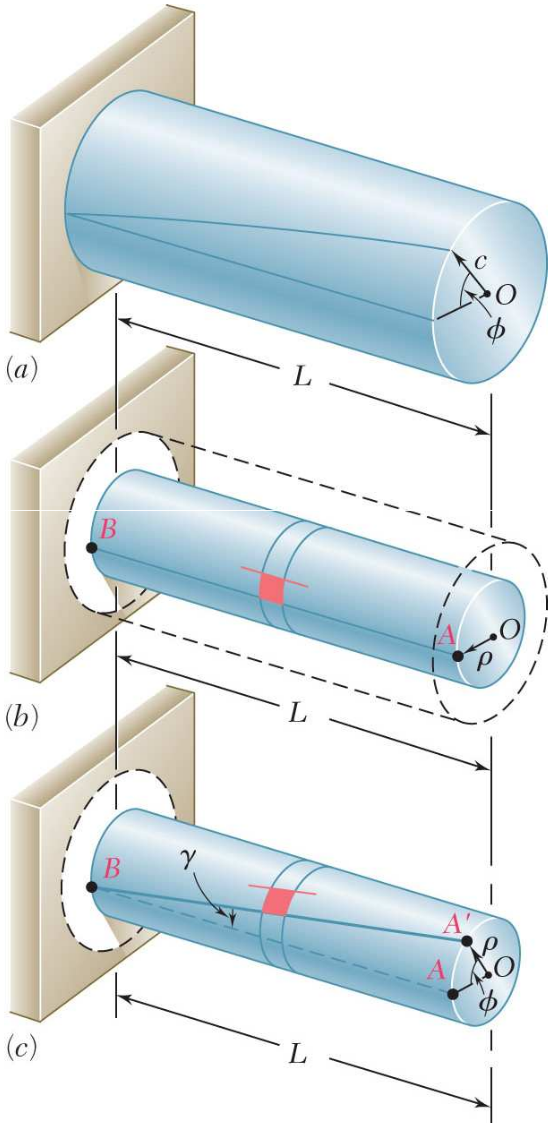
- From observation, the angle of twist of the shaft is proportional to the shaft length.

$$\phi \propto L$$



- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

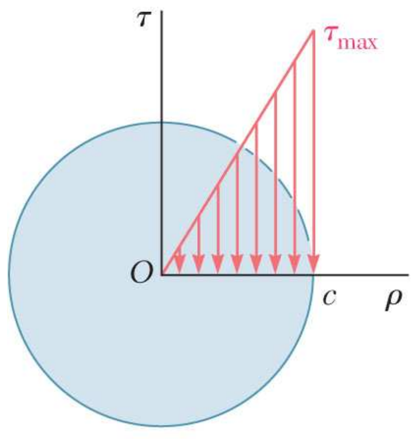
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

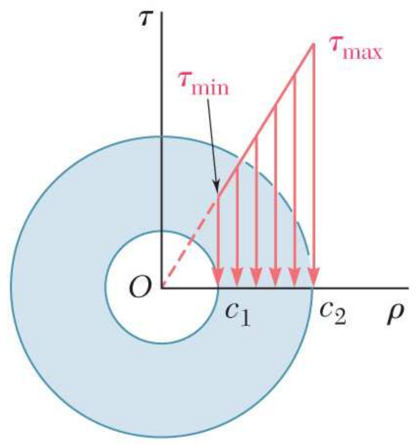
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range



(a)



(b)

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c}\tau_{max}$$

The shearing stress varies linearly with the radial position in the section.

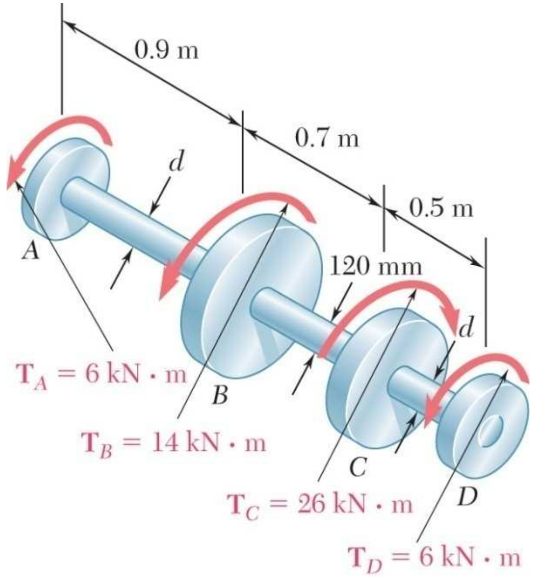
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau\, dA = \frac{\tau_{max}}{c}\int \rho^2\, dA = \frac{\tau_{max}}{c} J$$

- The results are known as the *elastic torsion formulas*,

$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Sample Problem 3.1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

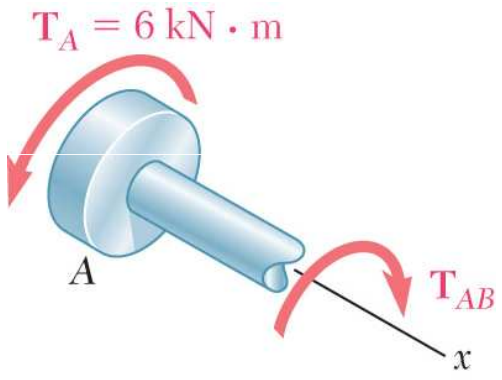
SOLUTION:

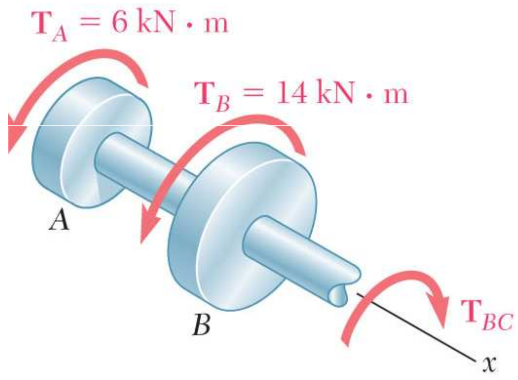
- Cut sections through shafts *AB*, *BC*, and *CD* and perform static equilibrium analyses to find internal torques.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter of *AB* and *CD*.

Sample Problem 3.1

SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.





$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$

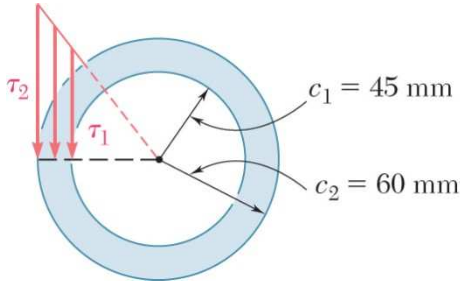
$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$

$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92\times 10^{-6}\,\text{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\text{kN}\cdot\text{m})(0.060\,\text{m})}{13.92\times 10^{-6}\,\text{m}^4}$$

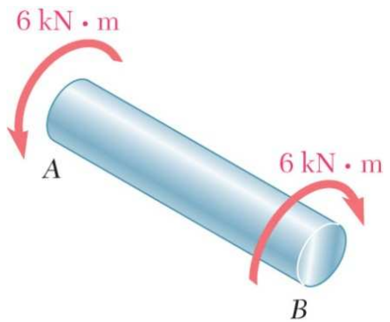
$$= 86.2\,\text{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\text{MPa}} = \frac{45\,\text{mm}}{60\,\text{mm}}$$

$$\tau_{min} = 64.7\,\text{MPa}$$

$$\tau_{max} = 86.2\,\text{MPa}$$
$$\tau_{min} = 64.7\,\text{MPa}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

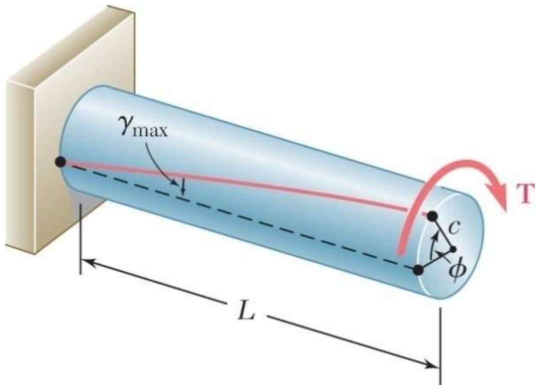


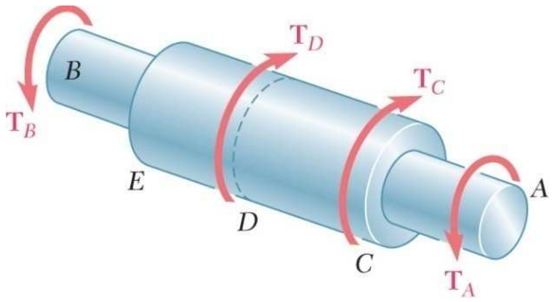
$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\text{kN}\cdot\text{m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times 10^{-3}\,\text{m}$$

$$d = 2c = 77.8\,\text{mm}$$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

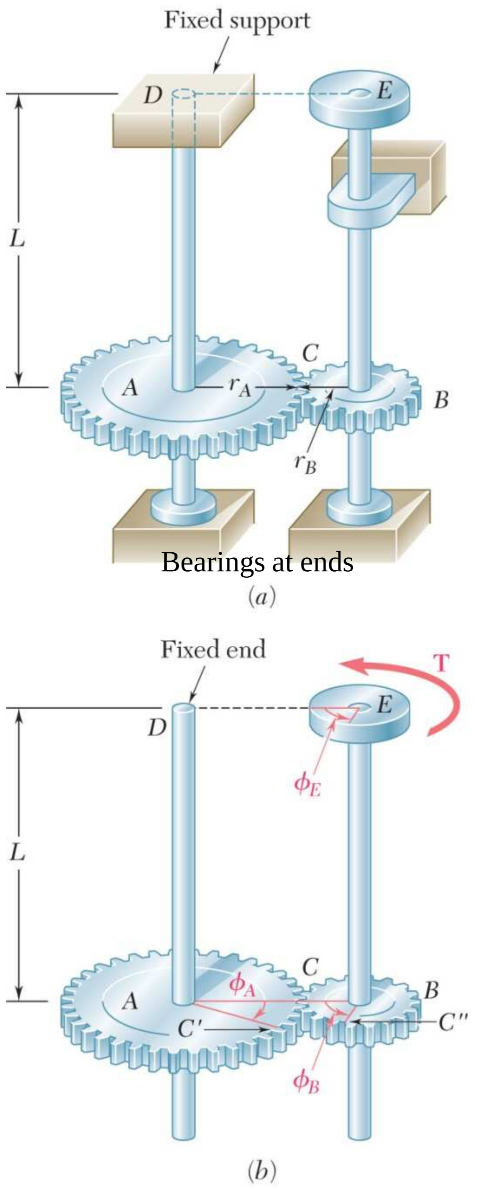
- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

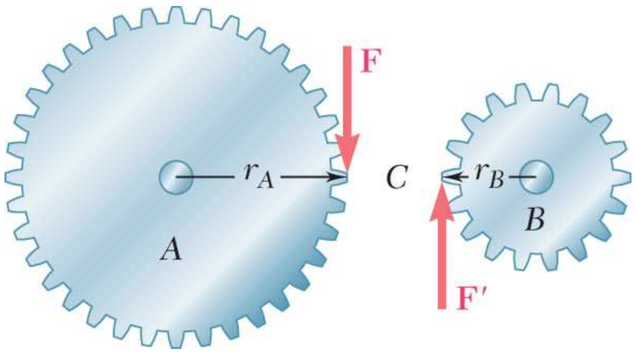
Geared Shafts



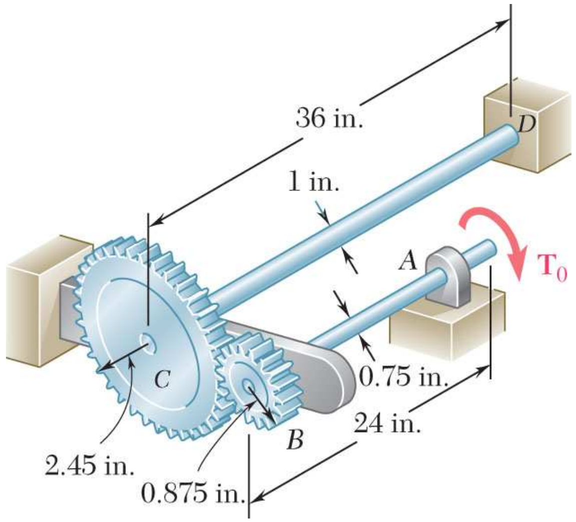
- For the assembly shown, knowing that

$$r_A = 2r_B$$

and the two shafts are identical (J, G) determine the angle of rotation of end *E* of shaft *BE* when the torque $\mathbf{T_E}$ is applied at *E*.



Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

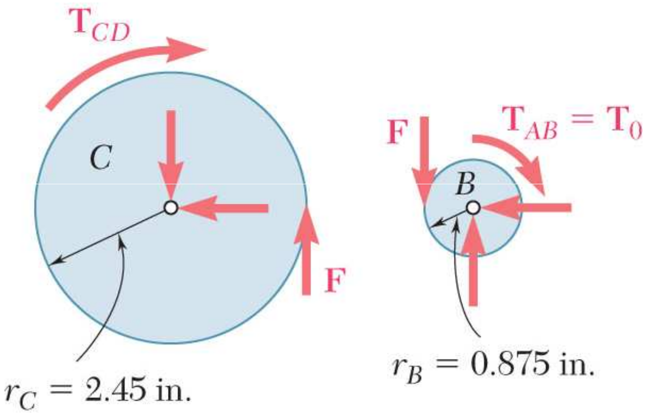
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0.
- Apply a kinematic analysis to relate the angular rotations of the gears.
- Find the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

Sample Problem 3.4

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .

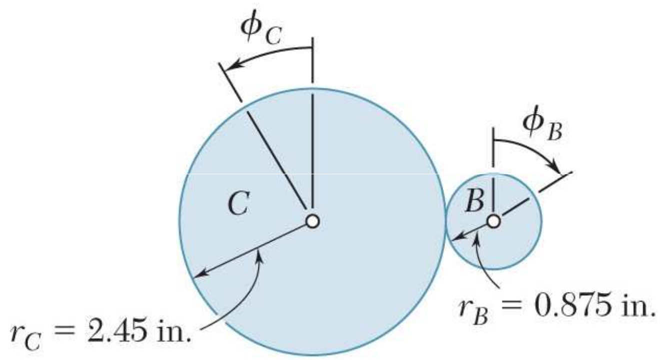


$$\sum M_B = 0 = F(0.875\text{in.}) - T_0$$

$$\sum M_C = 0 = F(2.45\text{in.}) - T_{CD}$$

$$T_{CD} = 2.8 T_0$$

- Apply a kinematic analysis to relate the angular rotations of the gears.



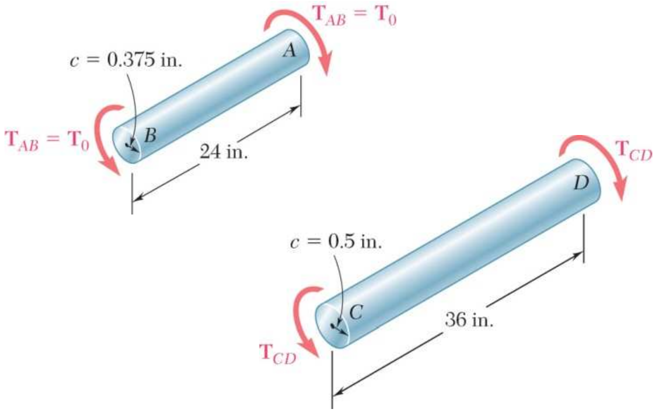
$$r_B \phi_B = r_C \phi_C$$

$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{2.45\text{in.}}{0.875\text{in.}}\phi_C$$

$$\phi_B = 2.8\phi_C$$

Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest.



$$\tau_{max} = \frac{T_{AB}c}{J_{AB}} \quad 8000\,psi = \frac{T_0(0.375\text{in.})}{\frac{\pi}{2}(0.375\text{in.})^4}$$

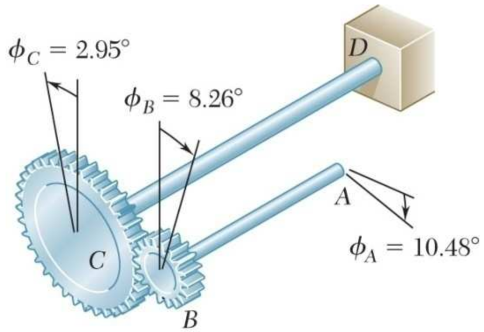
$$T_0 = 663\,\text{lb}\cdot\text{in.}$$

$$\tau_{max} = \frac{T_{CD}c}{J_{CD}} \quad 8000\,psi = \frac{2.8T_0(0.5\text{in.})}{\frac{\pi}{2}(0.5\text{in.})^4}$$

$$T_0 = 561\,\text{lb}\cdot\text{in.}$$

$$\boxed{T_0 = 561\,\text{lb}\cdot\text{in}}$$

- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.



$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(561\,\text{lb}\cdot\text{in.})(24\,\text{in.})}{\frac{\pi}{2}(0.375\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.387\,\text{rad} = 2.22^o$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.8(561\,\text{lb}\cdot\text{in.})(24\,\text{in.})}{\frac{\pi}{2}(0.5\text{in.})^4(11.2\times10^6\,\text{psi})}$$

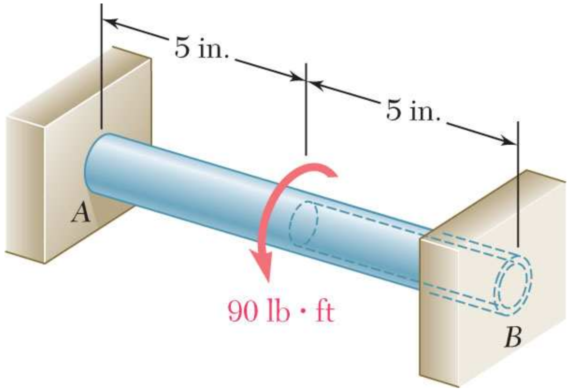
$$= 0.514\,\text{rad} = 2.95^o$$

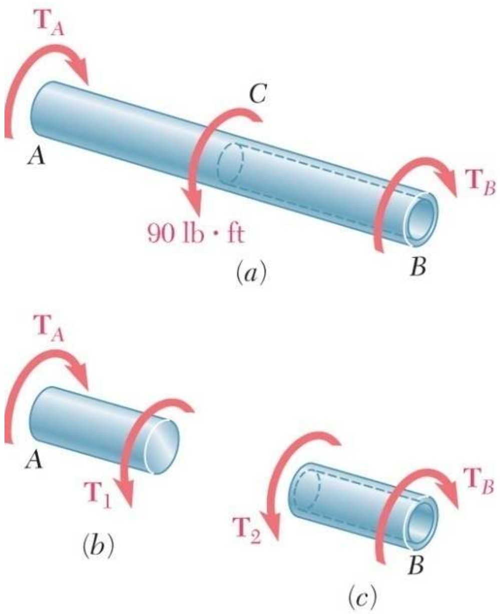
$$\phi_B = 2.8\phi_C = 2.8(2.95^o) = 8.26^o$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^o + 2.22^o$$

$$\boxed{\phi_A = 10.48^o}$$

Statically Indeterminate Shafts





- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

$$T_A + T_B = 90\,\text{lb}\cdot\text{ft}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90\,\text{lb}\cdot\text{ft}$$

Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed

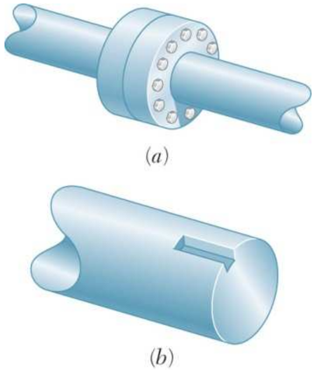
- Determine torque applied to shaft at specified power and speed,

$$P = T\omega$$

$$T = \frac{P}{\omega}$$

- Unit conversions
  - Power
    - 1 W = 1 Nm/s
    - 1 hp = 550 lb-ft/s
  - Speed
    - 1Hz = 1 rev/s = 2π rad/s
    - 1 rpm = 1 rev/min = 2π/60 rad/s

Stress Concentrations



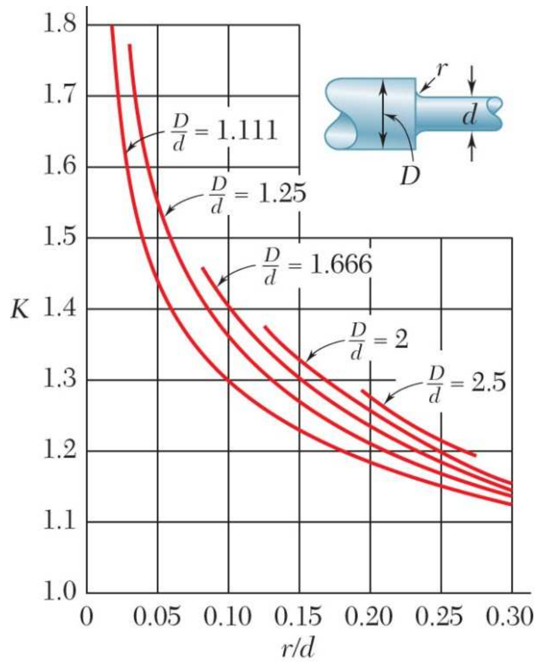


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$