CURVILINEAR MOTION: NORMAL AND TANGENTIAL COMPONENTS

Today's Objectives:

Students will be able to:

1. Determine the normal and tangential components of velocity and acceleration of a particle traveling along a curved path.

READING QUIZ

1. If a particle moves along a curve with a constant speed, then its tangential component of acceleration is

   A) positive. B) negative.

   C) zero. D) constant.

2. The normal component of acceleration represents

   A) the time rate of change in the magnitude of the velocity.

   B) the time rate of change in the direction of the velocity.

   C) magnitude of the velocity.

   D) direction of the total acceleration.

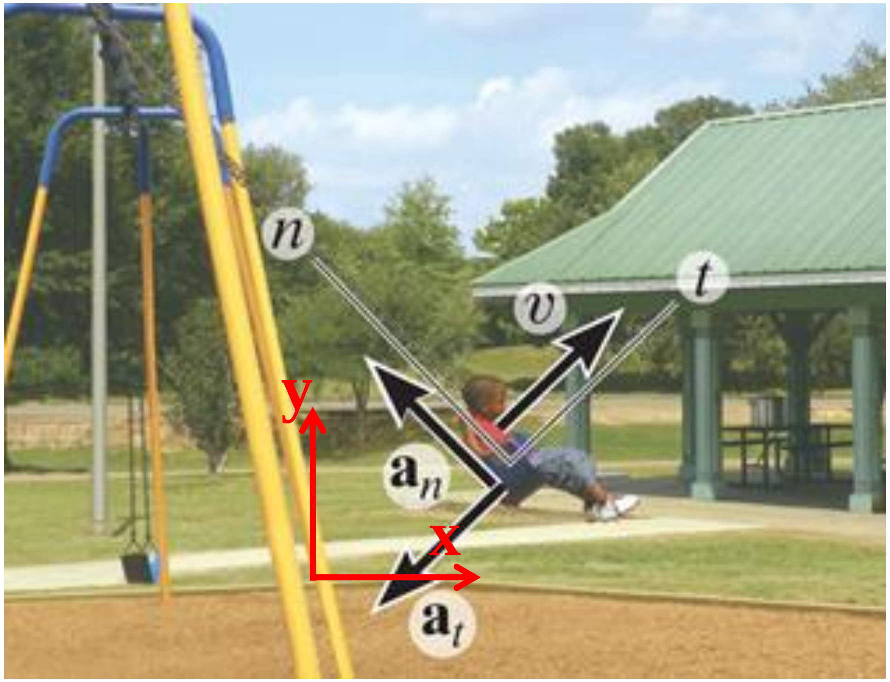
APPLICATIONS

Cars traveling along a clover-leaf interchange experience an acceleration due to a change in velocity as well as due to a change in direction of the velocity.

If the car's speed is increasing at a known rate as it travels along a curve, how can we determine the magnitude and direction of its total acceleration?

Why would you care about the total acceleration of the car? Rollover?

APPLICATIONS (continued)



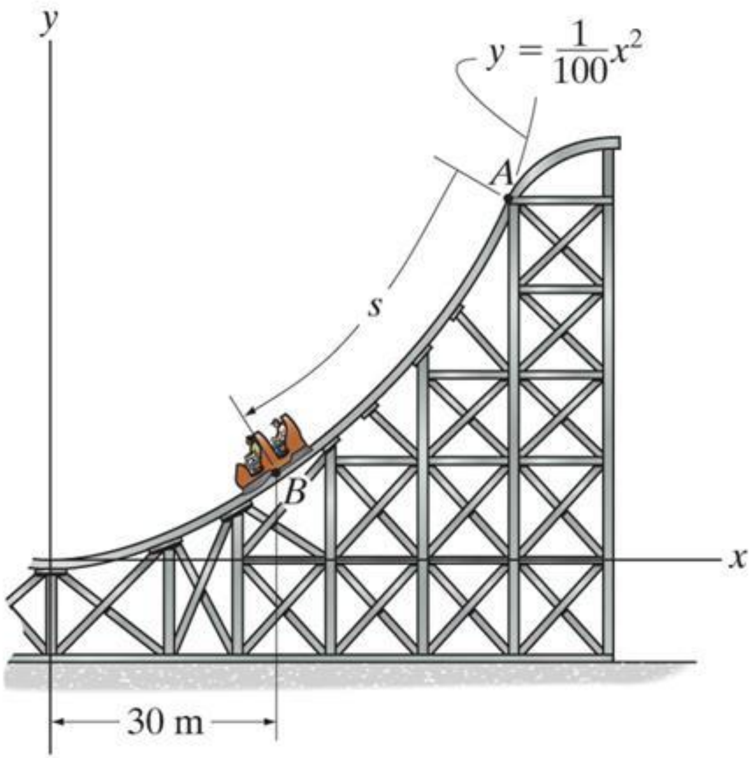
As the boy swings upward with a velocity **v**, his motion can be analyzed using *n–t* coordinates.

As he rises, the magnitude of his velocity is changing, and thus his acceleration is also changing.

How can we determine his velocity and acceleration at the bottom of the arc?

Can we use different coordinates, such as x-y coordinates, to describe his motion? Which coordinate system would be easier to use to describe his motion? Why?

APPLICATIONS (continued)



A roller coaster travels down a hill. The path is a function y = f(x).

The roller coaster starts from rest and increases its speed at a constant rate.

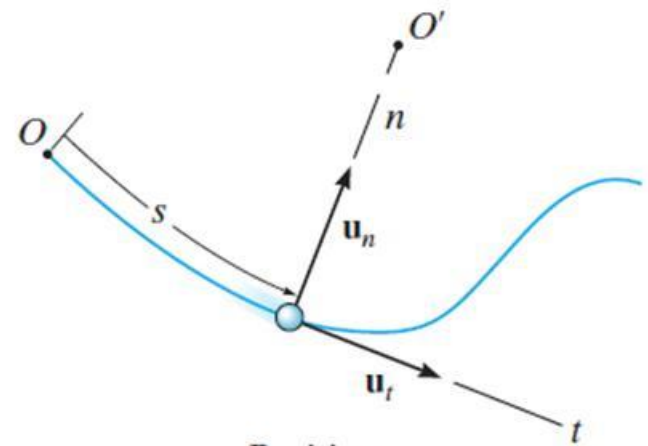
How can we determine its velocity and acceleration at the bottom?

Why would we want to know these values?

NORMAL AND TANGENTIAL COMPONENTS (Section 12.7)

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian (x-y-z). When the path of motion is known, normal (n) and tangential (t) coordinates are often used.

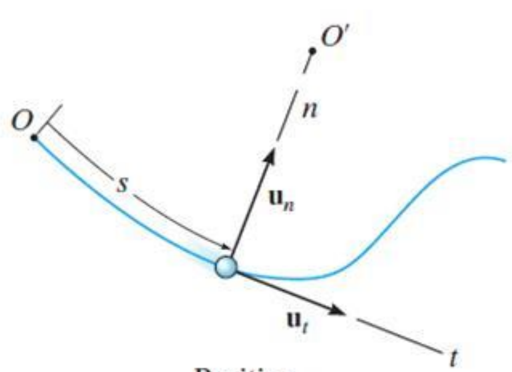
In the n-t coordinate system, the origin is located on the particle (thus the origin and coordinate system move with the particle).



Position

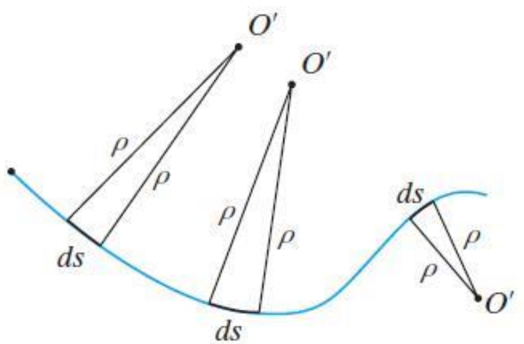
The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.
The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

NORMAL AND TANGENTIAL COMPONENTS (continued)



Position

The positive n and t directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively.

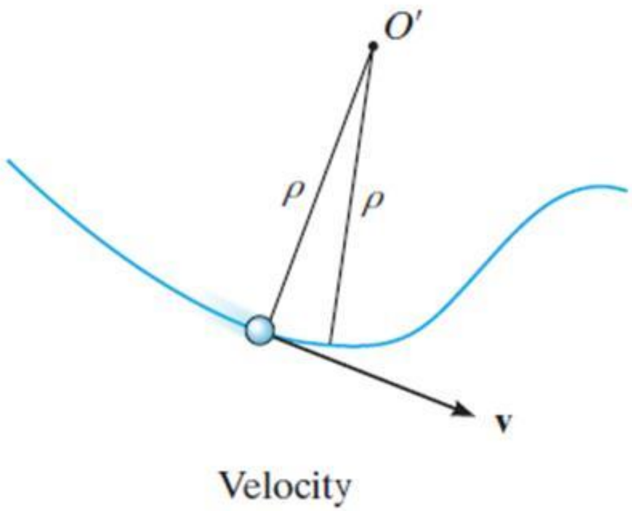


Radius of curvature

The center of curvature, O', always lies on the concave side of the curve. The radius of curvature, ρ, is defined as the perpendicular distance from the curve to the center of curvature at that point.

The position of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point.

VELOCITY IN THE n-t COORDINATE SYSTEM



The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, s(t).

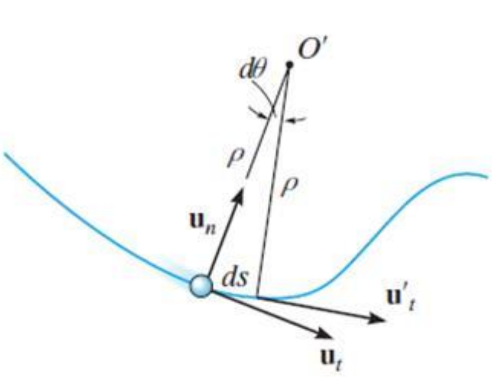
$$\boldsymbol{v} = v\,\boldsymbol{u}_t \quad \text{where} \quad v = \dot{s} = ds/dt$$

Here v defines the magnitude of the velocity (speed) and $\boldsymbol{u}_t$ defines the direction of the velocity vector.

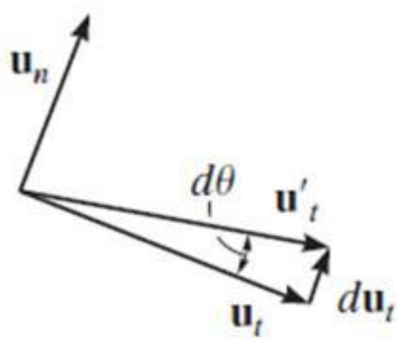
ACCELERATION IN THE n-t COORDINATE SYSTEM

Acceleration is the time rate of change of velocity:

$$\mathbf{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\boldsymbol{u}_t + v\dot{\boldsymbol{u}}_t$$



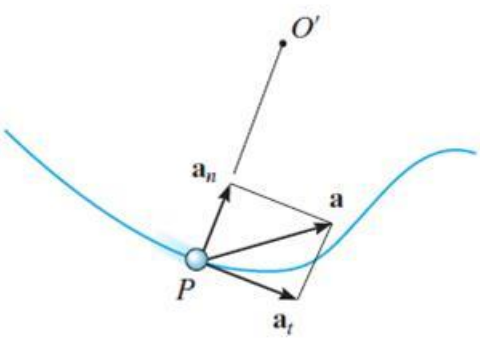
Here $\dot{v}$ represents the change in the magnitude of velocity and $\dot{\boldsymbol{u}}_t$ represents the rate of change in the direction of $\boldsymbol{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

$$\mathbf{a} = \dot{\mathbf{v}}\,\boldsymbol{u}_t + (v^2/\rho)\,\boldsymbol{u}_n = a_t\,\boldsymbol{u}_t + a_n\,\boldsymbol{u}_n.$$

ACCELERATION IN THE n-t COORDINATE SYSTEM (continued)



Acceleration

So, there are two components to the acceleration vector:

$$\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n$$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

$$a_t = \dot{v} \quad \text{or} \quad a_t \, ds = v \, dv$$

- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/\rho$

- The magnitude of the acceleration vector is

$$a = \sqrt{(a_n)^2 + (a_t)^2}$$

SPECIAL CASES OF MOTION

There are some special cases of motion to consider.

1) The particle moves along a straight line.

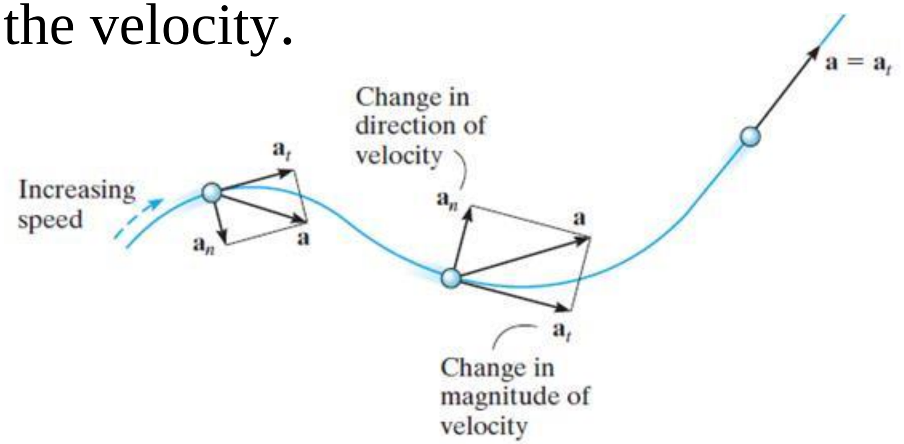
$\rho \longrightarrow \infty \quad => \quad a_n = v^2/\rho = 0 \quad => \quad a = a_t = \dot{v}$

The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

$a_t = \dot{v} = 0 \quad => \quad a = a_n = v^2/\rho$

The normal component represents the time rate of change in the direction of the velocity.



SPECIAL CASES OF MOTION (continued)

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$.
In this case,

$$s = s_o + v_o t + (1/2)\,(a_t)_c\, t^2$$

$$v = v_o + (a_t)_c\, t$$

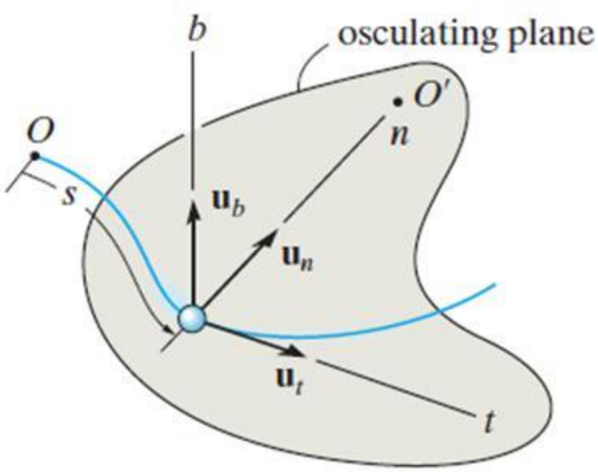
$$v^2 = (v_o)^2 + 2\,(a_t)_c\,(s - s_o)$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$. How are these equations related to projectile motion equations? Why?

4) The particle moves along a path expressed as $y = f(x)$.
The radius of curvature, ρ, at any point on the path can be calculated from

$$\rho = \frac{[\,1 + (dy/dx)^2\,]^{3/2}}{\left| d^2y/dx^2 \right|}$$

THREE-DIMENSIONAL MOTION

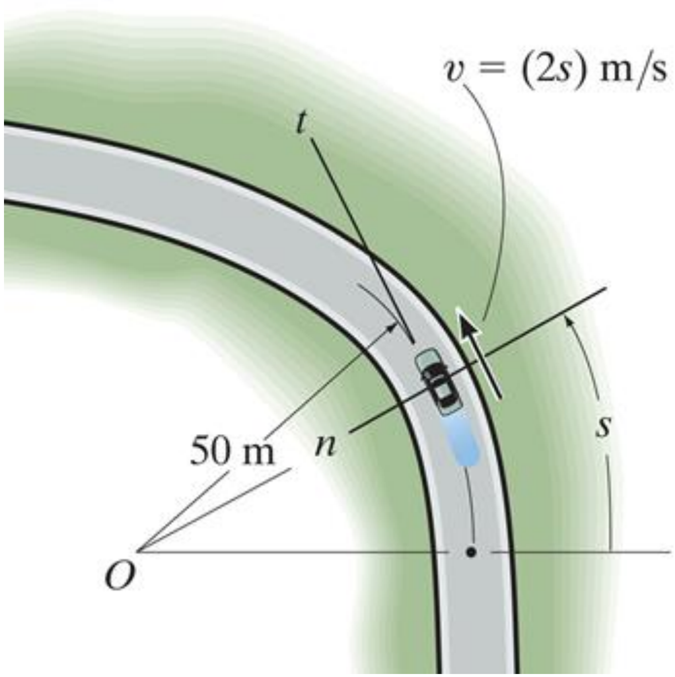


If a particle moves along a space curve, the n-t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n-t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\boldsymbol{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product $\boldsymbol{u}_b = \boldsymbol{u}_t \times \boldsymbol{u}_n$.

There is no motion, thus no velocity or acceleration, in the binomial direction.

EXAMPLE I



Given: A car travels along the road with a speed of v = (2s) m/s, where s is in meters.
ρ = 50 m

Find: The magnitudes of the car's acceleration at s = 10 m.

Plan:

1) Calculate the velocity when s = 10 m using v(s).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

EXAMPLE I (continued)

Solution:

1) The velocity vector is $\boldsymbol{v} = v\,\boldsymbol{u}_t$, where the magnitude is given by v = (2s) m/s.

   When s = 10 m: v = 20 m/s

2) The acceleration vector is $\mathbf{a} = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n = \dot{v}\boldsymbol{u}_t + (v^2/\rho)\boldsymbol{u}_n$

   Tangential component:

   Since $a_t = \dot{v} = dv/dt = (dv/ds)\,(ds/dt) = v\,(dv/ds)$

   where v = 2s $\Rightarrow$ $a_t = d(2s)/ds\,(v) = 2\,v$

   At s = 10 m: $a_t = 40\text{ m/s}^2$

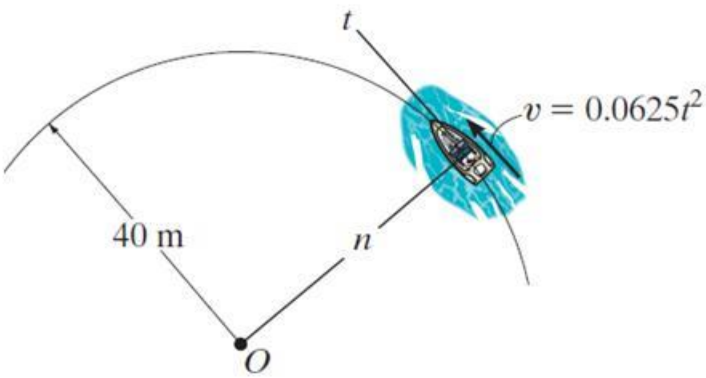
   Normal component: $a_n = v^2/\rho$

   When s = 10 m: $a_n = (20)^2 / (50) = 8\text{ m/s}^2$

   The magnitude of the acceleration is

   $a = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{40^2 + 8^2} = \underline{40.8\text{ m/s}^2}$

EXAMPLE II



Given: A boat travels around a circular path, ρ = 40 m, at a speed that increases with time, $v = (0.0625\ t^2)$ m/s.

Find: The magnitudes of the boat's velocity and acceleration at the instant t = 10 s.

Plan:

The boat starts from rest (v = 0 when t = 0).

1) Calculate the velocity at t = 10 s using v(t).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

EXAMPLE II (continued)

Solution:

1) The velocity vector is $\mathbf{v} = v\,\mathbf{u}_t$, where the magnitude is given by $v = (0.0625t^2)$ m/s. At t = 10s:

$$v = 0.0625\,t^2 = 0.0625\,(10)^2 = \underline{6.25\text{ m/s}}$$

2) The acceleration vector is $\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n = \dot{v}\mathbf{u}_t + (v^2/\rho)\mathbf{u}_n$.

Tangential component: $a_t = \dot{v} = d(.0625\,t^2)/dt = 0.125\,t \text{ m/s}^2$

At t = 10s: $a_t = 0.125t = 0.125(10) = 1.25\text{ m/s}^2$

Normal component: $a_n = v^2/\rho \;\; \text{m/s}^2$

At t = 10s: $a_n = (6.25)^2 / (40) = 0.9766\text{ m/s}^2$

The magnitude of the acceleration is

$$a = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{1.25^2 + 0.9766^2} = \underline{1.59\text{ m/s}^2}$$

CONCEPT QUIZ

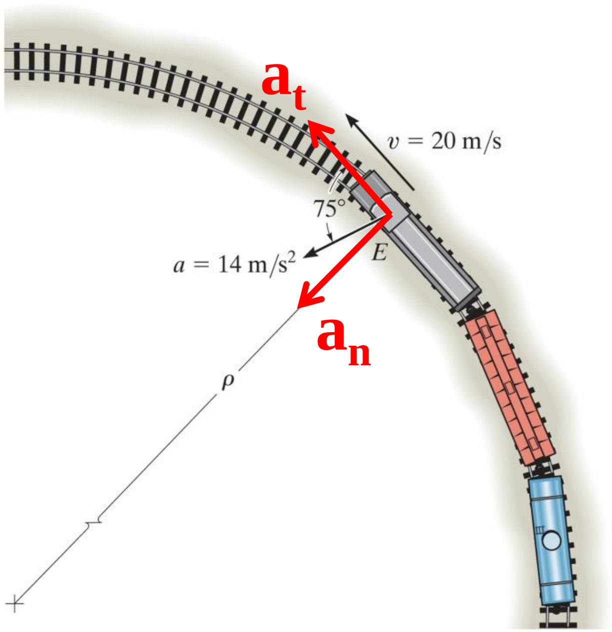
1. A particle traveling in a circular path of radius 300 m has an instantaneous velocity of 30 m/s and its velocity is increasing at a constant rate of 4 m/s^2. What is the magnitude of its total acceleration at this instant?

   A) 3 m/s^2
   B) 4 m/s^2
   C) 5 m/s^2
   D) -5 m/s^2

2. If a particle moving in a circular path of radius 5 m has a velocity function $v = 4t^2$ m/s, what is the magnitude of its total acceleration at t = 1 s?

   A) 8 m/s
   B) 8.6 m/s
   C) 3.2 m/s
   D) 11.2 m/s

GROUP PROBLEM SOLVING I



Given: The train engine at E has a speed of 20 m/s and an acceleration of 14 m/s^2 acting in the direction shown.

Find: The rate of increase in the train's speed and the radius of curvature ρ of the path.

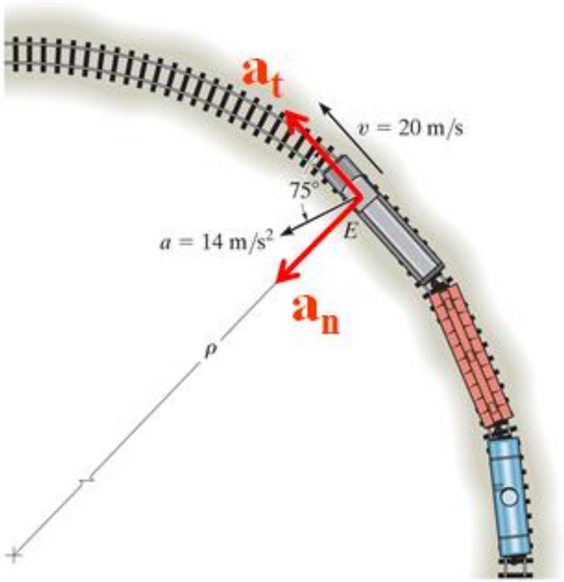
Plan:

1. Determine the tangential and normal components of the acceleration.
2. Calculate $\dot{v}$ from the tangential component of the acceleration.
3. Calculate ρ from the normal component of the acceleration.

GROUP PROBLEM SOLVING I (continued)

Solution:

1) Acceleration



Tangential component :

$a_t = 14 \cos(75) = 3.623 \text{ m/s}^2$

Normal component :

$a_n = 14 \sin(75) = 13.52 \text{ m/s}^2$

2) The tangential component of acceleration is the rate of increase of the train's speed, so

$a_t = \dot{v} = \underline{3.62 \text{ m/s}^2}$.

3) The normal component of acceleration is

$a_n = v^2/\rho \quad \Rightarrow 13.52 = 20^2 / \rho$

$\rho = \underline{29.6 \text{ m}}$

GROUP PROBLEM SOLVING II

Given: Starting from rest, a bicyclist travels around a horizontal circular path, $\rho = 10$ m, at a speed of $v = (0.09\ t^2 + 0.1\ t)$ m/s.

Find: The magnitudes of her velocity and acceleration when she has traveled 3 m.

Plan:

The bicyclist starts from rest ($v = 0$ when $t = 0$).

1) Integrate v(t) to find the position s(t).
2) Calculate the time when s = 3 m using s(t).
3) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

GROUP PROBLEM SOLVING II (continued)

Solution:

1) The velocity vector is $v = (0.09\ t^2 + 0.1\ t)$ m/s, where t is in seconds. Integrate the velocity and find the position s(t).

Position: $\int v\,dt = \int (0.09\ t^2 + 0.1\ t)\,dt$

$$s(t) = 0.03\ t^3 + 0.05\ t^2$$

2) Calculate the time, t when s = 3 m.

$$3 = 0.03\ t^3 + 0.05\ t^2$$

Solving for t, $t = 4.147$ s

The velocity at t = 4.147 s is,

$$v = 0.09\ (4.147)^2 + 0.1\ (4.147) = \underline{1.96\ \text{m/s}}$$

GROUP PROBLEM SOLVING II (continued)

3) The acceleration vector is $\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n = \dot{v} \boldsymbol{u}_t + (v^2/\rho) \boldsymbol{u}_n$.

Tangential component:

$$a_t = \dot{v} = d(0.09\, t^2 + 0.1\, t) / dt = (0.18\, t + 0.1) \text{ m/s}^2$$

At t = 4.147 s : $a_t = 0.18\,(4.147) + 0.1 = 0.8465 \text{ m/s}^2$

Normal component:

$$a_n = v^2/\rho \ \text{ m/s}^2$$

At t = 4.147 s : $a_n = (1.96)^2 / (10) = 0.3852 \text{ m/s}^2$

The magnitude of the acceleration is

$$a = \sqrt{(a_n)^2 + (a_t)^2} = \sqrt{0.8465^2 + 0.3852^2} = 0.930 \text{ m/s}^2$$

ATTENTION QUIZ

1. The magnitude of the normal acceleration is

   A) proportional to radius of curvature.

   B) inversely proportional to radius of curvature.

   C) sometimes negative.

   D) zero when velocity is constant.

2. The directions of the tangential acceleration and velocity are always

   A) perpendicular to each other.

   B) collinear.

   C) in the same direction.

   D) in opposite directions.

End of the Lecture

Let Learning Continue