MAE 322
Machine Design
Lecture 5
Fatigue - 3

Dr. Hodge Jenkins
Mercer University
Returning to Stress-Life Fatigue Modeling

Fatigue Stress-Life: $S_f$-$N$ Diagram for steels

- Stress levels below $S_e$ (Endurance Strength) predict infinite life.
- Between $10^3$ and $10^6$ cycles, finite life is predicted.
- Below $10^3$ cycles is known as low cycle.

Fig. 6–10
Stress Concentration and Notch Sensitivity

- For dynamic loading, stress concentration effects must be applied.
- Obtain the Stress Concentration factor $K_t$ as usual (e.g. Appendix A–15)
- In fatigue, some materials are not fully sensitive to $K_t$ so a reduced value can be used. More ductile => less sensitive.
- Define $K_f$ as the *fatigue stress-concentration factor*.
- Define $q$ as *notch sensitivity*, ranging from 0 (not sensitive) to 1 (fully sensitive).

$$q = \frac{K_f - 1}{K_t - 1} \quad \text{or} \quad q_{\text{shear}} = \frac{K_{fs} - 1}{K_{ts} - 1} \quad (6-31)$$

- For $q = 0$, $K_f = 1$
- For $q = 1$, $K_f = K_t$

$$\sigma = K_f \sigma_{\text{nominal}}$$
Notch Sensitivity

- Obtain \( q \) for bending or axial loading from Fig. 6–20.
- Then get \( K_f \) from Eq. (6–32):
  \[
  K_f = 1 + q(K_t - 1)
  \]
Notch Sensitivity

- Obtain $q_s$ for torsional loading from Fig. 6–21.
- Then get $K_{fs}$ from Eq. (6–32): $K_{fs} = 1 + q_s (K_{ts} - 1)$
Notch Sensitivity for Cast Irons

- Cast irons are already full of discontinuities, which are included in the strengths.
- Additional notches do not add much additional harm.
- Recommended to use $q = 0.2$ for cast irons.
Example 6–6

A steel shaft in bending has an ultimate strength of 690 MPa and a shoulder with a fillet radius of 3 mm connecting a 32-mm diameter with a 38-mm diameter. Estimate $K_f$ using (a) Figure 6–20.

Solution

From Fig. A–15–9, using $D/d = 38/32 = 1.1875$, $r/d = 3/32 = 0.09375$, we read the graph to find $K_t = 1.65$.

(a) From Fig. 6–20, for $S_{ut} = 690$ MPa and $r = 3$ mm, $q = 0.84$. Thus, from Eq. (6–32)

$$K_f = 1 + q(K_t - 1) = 1 + 0.84(1.65 - 1) = 1.55$$

Answer
Application of Fatigue Stress Concentration Factor

- Use $K_f$ as a multiplier to increase the nominal stress.
- For infinite life, either method is equivalent, since

$$n_f = \frac{S_e}{K_f \sigma} = \frac{1}{K_f} \frac{S_e}{\sigma}$$

- For finite life, increasing stress is more conservative. Decreasing $S_e$ applies more to high cycle than low cycle.
- $\sigma$ is pure alternating stress, in this case.
Example 6–7

For the step-shaft of Ex. 6–6, it is determined that the fully corrected endurance limit is $S_e = 280$ MPa. Consider the shaft undergoes a fully reversing nominal stress in the fillet of $(\sigma_{rev})_{nom} = 260$ MPa. Estimate the number of cycles to failure.

**Solution**

From Ex. 6–6, $K_f = 1.55$, and the ultimate strength is $S_{ut} = 690$ MPa $= 100$ kpsi. The maximum reversing stress is

$$(\sigma_{rev})_{max} = K_f (\sigma_{rev})_{nom} = 1.55(260) = 403$$

From Fig. 6–18, $f = 0.845$. From Eqs. (6–14), (6–15), and (6–16)

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{[0.845(690)]^2}{280} = 1214 \text{ MPa}$$

$$b = -\frac{1}{3} \log \frac{f S_{ut}}{S_e} = -\frac{1}{3} \log \left[ \frac{0.845(690)}{280} \right] = -0.1062$$

$$N = \left( \frac{\sigma_{rev}}{a} \right)^{1/b} = \left( \frac{403}{1214} \right)^{1/-0.1062} = 32.3(10^3) \text{ cycles} \quad \text{Answer}$$
Example 6–8

A 1015 hot-rolled steel bar has been machined to a diameter of 1 in. It is to be placed in reversed axial loading for 70,000 cycles to failure in an operating environment of 550°F. Using ASTM minimum properties, and a reliability of 99 percent, estimate the endurance limit and fatigue strength at 70,000 cycles.

Solution

From Table A–20, $S_{ut} = 50$ kpsi at 70°F. Since the rotating-beam specimen endurance limit is not known at room temperature, we determine the ultimate strength at the elevated temperature first, using Table 6–4. From Table 6–4,

$$
\left( \frac{S_T}{S_{RT}} \right)_{550^\circ} = \frac{0.995 + 0.963}{2} = 0.979
$$

The ultimate strength at 550°F is then

$$(S_{ut})_{550^\circ} = (S_T/S_{RT})_{550^\circ} (S_{ut})_{70^\circ} = 0.979(50) = 49.0 \text{ kpsi}$$

The rotating-beam specimen endurance limit at 550°F is then estimated from Eq. (6–8) as

$$S'_e = 0.5(49) = 24.5 \text{ kpsi}$$
Example 6–8 (continued)

Next, we determine the Marin factors. For the machined surface, Eq. (6–19) with Table 6–2 gives

\[ k_a = aS_{ut}^b = 2.70(49^{-0.265}) = 0.963 \]

For axial loading, from Eq. (6–21), the size factor \( k_b = 1 \), and from Eq. (6–26) the loading factor is \( k_c = 0.85 \). The temperature factor \( k_d = 1 \), since we accounted for the temperature in modifying the ultimate strength and consequently the endurance limit. For 99 percent reliability, from Table 6–5, \( k_e = 0.814 \). Finally, since no other conditions were given, the miscellaneous factor is \( k_f = 1 \). The endurance limit for the part is estimated by Eq. (6–18) as

\[ S_e = k_a k_b k_c k_d k_e k_f S'_e \]

\[ = 0.963(1)(0.85)(1)(0.814)(1)24.5 = 16.3 \text{ kpsi} \]

Answer
For the fatigue strength at 70,000 cycles we need to construct the S-N equation. From p. 293, since $S_{ut} = 49 < 70$ kpsi, then $f = 0.9$. From Eq. (6–14)

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{[0.9(49)]^2}{16.3} = 119.3 \text{ kpsi}$$

and Eq. (6–15)

$$b = -\frac{1}{3} \log \left( \frac{f S_{ut}}{S_e} \right) = -\frac{1}{3} \log \left( \frac{0.9(49)}{16.3} \right) = -0.1441$$

Finally, for the fatigue strength at 70,000 cycles, Eq. (6–13) gives

$$S_f = a N^b = 119.3(70,000)^{-0.1441} = 23.9 \text{ kpsi} \quad \text{Answer}$$
Characterizing Fluctuating Stresses

- The $S-N$ diagram is applicable for *completely reversed* stresses.
- Other fluctuating stresses exist.
- Sinusoidal loading patterns are common, but not necessary.
Fluctuating Stresses

- General Fluctuating
- Repeated
- Completely Reversed

Fig. 6–23
Characterizing Fluctuating Stresses

- Fluctuating stresses can often be characterized simply by the minimum and maximum stresses, $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$.

- Define $\sigma_m$ as midrange steady component of stress (sometimes called mean stress) and $\sigma_a$ as amplitude of alternating component of stress.

\[
\sigma_m = \frac{\sigma_{\text{max}} + \sigma_{\text{min}}}{2}
\]

\[
\sigma_a = \left| \frac{\sigma_{\text{max}} - \sigma_{\text{min}}}{2} \right|
\]

(6–36)
Characterizing Fluctuating Stresses

- Other useful definitions include *stress ratio*

\[ R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \]  

*(6-37)*

and *amplitude ratio*

\[ A = \frac{\sigma_a}{\sigma_m} \]  

*(6-38)*
Application of $K_f$ for Fluctuating Stresses

- For fluctuating loads at points with stress concentration, the best approach is to design to avoid all localized plastic strain.
- In this case, $K_f$ should be applied to both alternating and midrange stress components.
- When localized strain does occur, some methods (e.g. nominal mean stress method and residual stress method) recommend only applying $K_f$ to the alternating stress.
- The *Dowling method* recommends applying $K_f$ to the alternating stress and $K_{fm}$ to the mid-range stress, where $K_{fm}$ is

\[
K_{fm} = K_f \\
K_{fm} = \frac{S_y - K_f \sigma_{ao}}{|\sigma_{mo}|} \\
K_{fm} = 0
\]

\[
K_f |\sigma_{max,o}| < S_y \\
K_f |\sigma_{max,o}| > S_y \\
K_f |\sigma_{max,o} - \sigma_{min,o}| > 2S_y
\] (6-39)
Fatigue Failure for Fluctuating Stresses

- Vary the $\sigma_m$ and $\sigma_a$ to learn about the fatigue resistance under fluctuating loading
- Three common methods of plotting results follow.
Plot of Alternating vs Midrange Stress

- Probably most common and simple to use is the plot of $\sigma_a$ vs $\sigma_m$
- Has gradually usurped the name of Goodman or Modified Goodman diagram
- Modified Goodman line from $S_e$ to $S_{ut}$ is one simple representation of the limiting boundary for infinite life
Plot of Alternating vs Midrange Stress

- Experimental data on normalized plot of $\sigma_a$ vs $\sigma_m$
- Demonstrates little effect of negative midrange stress
Commonly Used Failure Criteria

- Five commonly used failure criteria are shown
- Gerber passes through the data (best Fit)
- ASME-elliptic passes through data and incorporates rough yielding check

Fig. 6–27

Shigley’s Mechanical Engineering Design
Commonly Used Failure Criteria

- Modified Goodman is linear, so simple to use for design. It is more conservative than Gerber.
- Soderberg provides a very conservative single check of both fatigue and yielding.

Fig. 6–27

Shigley’s Mechanical Engineering Design
Commonly Used Failure Criteria

- **Langer line** represents standard yield check.
- It is equivalent to comparing maximum stress to yield strength.

\[ S_y \]

\[ S_e \]

\[ S_m \]

\[ S_a \]

\[ S_{ut} \]

Fig. 6–27
Equations for Commonly Used Failure Criteria

- Intersecting a constant slope load line with each failure criteria produces design equations
- \( n \) is the design factor or factor of safety for infinite fatigue life

\[
\text{Soderberg} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \quad (6-45)
\]

\[
\text{mod-Goodman} \quad \frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \quad (6-46)
\]

\[
\text{Gerber} \quad \frac{n\sigma_a}{S_e} + \left( \frac{n\sigma_m}{S_{ut}} \right)^2 = 1 \quad (6-47)
\]

\[
\text{ASME-elliptic} \quad \left( \frac{n\sigma_a}{S_e} \right)^2 + \left( \frac{n\sigma_m}{S_y} \right)^2 = 1 \quad (6-48)
\]
Modified Goodman Fatigue Failure Theory

Fatigue factor of safety

\[ n_f = \frac{1}{\sigma_a \frac{\sigma_m}{S_e} \frac{S_{ut}}{S_e}} \]

ASME-Elliptic Fatigue Failure Theory

Fatigue factor of safety

\[ n_f = \sqrt{\frac{1}{(\sigma_a/S_e)^2 + (\sigma_m/S_y)^2}} \]
Gerber Fatigue Failure Theory

Fatigue factor of safety

\[ n_f = \frac{1}{2} \left( \frac{S_{ut}}{\sigma_m} \right)^2 \frac{\sigma_a}{S_e} \left[ -1 + \sqrt{1 + \left( \frac{2\sigma_m S_e}{S_{ut} \sigma_a} \right)^2} \right] \quad \sigma_m > 0 \]
Example 6–10

A 1.5-in-diameter bar has been machined from an AISI 1050 cold-drawn bar. This part is to withstand a fluctuating tensile load varying from 0 to 16 kip. Because of the ends, and the fillet radius, a fatigue stress-concentration factor $K_f$ is 1.85 for $10^6$ or larger life. Find $S_a$ and $S_m$ and the factor of safety guarding against fatigue and first-cycle yielding, using (a) the Gerber fatigue line and (b) the ASME-elliptic fatigue line.

Solution

We begin with some preliminaries. From Table A–20, $S_{ut} = 100$ kpsi and $S_y = 84$ kpsi. Note that $F_a = F_m = 8$ kip. The Marin factors are, deterministically,

$$k_a = 2.70(100)^{-0.265} = 0.797: \text{Eq. (6–19), Table 6–2, p. 296}$$

$$k_b = 1 \text{ (axial loading, see } k_c)$$

$$k_c = 0.85: \text{Eq. (6–26), p. 298}$$

$$k_d = k_e = k_f = 1$$

$$S_e = 0.797(1)0.850(1)(1)(1)0.5(100) = 33.9 \text{ kpsi: Eqs. (6–8), (6–18), p. 290, p. 295}$$
Example 6–10 (continued)

The nominal axial stress components $\sigma_{ao}$ and $\sigma_{mo}$ are

$$\sigma_{ao} = \frac{4F_a}{\pi d^2} = \frac{4(8)}{\pi (1.5)^2} = 4.53 \text{ kpsi} \quad \sigma_{mo} = \frac{4F_m}{\pi d^2} = \frac{4(8)}{\pi (1.5)^2} = 4.53 \text{ kpsi}$$

Applying $K_f$ to both components $\sigma_{ao}$ and $\sigma_{mo}$ constitutes a prescription of no notch yielding:

$$\sigma_a = K_f \sigma_{ao} = 1.85(4.53) = 8.38 \text{ kpsi} = \sigma_m$$

(a) Let us calculate the factors of safety first. From the bottom panel from Table 6–7 the factor of safety for fatigue is

$$n_f = \frac{1}{2} \left( \frac{100}{8.38} \right)^2 \left( \frac{8.38}{33.9} \right) \left\{ -1 + \sqrt{1 + \left[ \frac{2(8.38)33.9}{100(8.38)} \right]^2} \right\} = 3.66$$

From Eq. (6–49) the factor of safety guarding against first-cycle yield is

$$n_y = \frac{S_y}{\sigma_a + \sigma_m} = \frac{84}{8.38 + 8.38} = 5.01$$
Thus, we see that fatigue will occur first and the factor of safety is 3.68. This can be seen in Fig. 6–28 where the load line intersects the Gerber fatigue curve first at point $B$. If the plots are created to true scale it would be seen that $n_f = OB/OA$. 

Fig. 6–28
Example 6–10 (continued)

From the first panel of Table 6–7, \( r = \sigma_a / \sigma_m = 1 \),

\[
S_a = \frac{(1)^2 100^2}{2(33.9)} \left\{ -1 + \sqrt{1 + \left[ \frac{2(33.9)}{(1)100} \right]^2} \right\} = 30.7 \text{ kpsi} \quad \text{Answer}
\]

\[
S_m = \frac{S_a}{r} = \frac{30.7}{1} = 30.7 \text{ kpsi} \quad \text{Answer}
\]

As a check on the previous result, \( n_f = \frac{OB}{OA} = \frac{S_a}{\sigma_a} = \frac{S_m}{\sigma_m} = \frac{30.7}{8.38} = 3.66 \) and we see total agreement.

Fig. 6–28
Example 6–10 (continued)

We could have detected that fatigue failure would occur first without drawing Fig. 6–28 by calculating \( r_{crit} \). From the third row third column panel of Table 6–7, the intersection point between fatigue and first-cycle yield is

\[
S_m = \frac{100^2}{2(33.9)} \left[ 1 - \sqrt{1 + \left( \frac{2(33.9)}{100} \right)^2 \left( 1 - \frac{84}{33.9} \right)} \right] = 64.0 \text{ kpsi}
\]

\[
S_a = S_y - S_m = 84 - 64 = 20 \text{ kpsi}
\]

The critical slope is thus

\[
r_{crit} = \frac{S_a}{S_m} = \frac{20}{64} = 0.312
\]

which is less than the actual load line of \( r = 1 \). This indicates that fatigue occurs before first-cycle-yield.
Example 6–10 (continued)

(b) Repeating the same procedure for the ASME-elliptic line, for fatigue

\[ n_f = \sqrt{\frac{1}{(8.38/33.9)^2 + (8.38/84)^2}} = 3.75 \quad \text{Answer} \]

Again, this is less than \( n_y = 5.01 \) and fatigue is predicted to occur first. From the first row second column panel of Table 6–8, with \( r = 1 \), we obtain the coordinates \( S_a \) and \( S_m \) of point \( B \) in Fig. 6–29 as

\[ S_a = \sqrt{\frac{(1)^2 33.9^2 (84)^2}{33.9^2 + (1)^2 84^2}} = 31.4 \text{ kpsi}, \]

\[ S_m = \frac{S_a}{r} = \frac{31.4}{1} = 31.4 \text{ kpsi} \quad \text{Answer} \]

To verify the fatigue factor of safety,

\[ n_f = \frac{S_a}{\sigma_a} = \frac{31.4}{8.38} = 3.75. \]
As before, let us calculate $r_{\text{crit}}$. From the third row second column panel of Table 6–8,

\[ S_a = \frac{2(84)33.9^2}{33.9^2 + 84^2} = 23.5 \text{ kpsi}, \quad S_m = S_y - S_a = 84 - 23.5 = 60.5 \text{ kpsi} \]

\[ r_{\text{crit}} = \frac{S_a}{S_m} = \frac{23.5}{60.5} = 0.388 \]

which again is less than $r = 1$, verifying that fatigue occurs first with $n_f = 3.75$.

The Gerber and the ASME-elliptic fatigue failure criteria are very close to each other and are used interchangeably. The ANSI/ASME Standard B106.1M–1985 uses ASME-elliptic for shafting.
Example 6–12

A steel bar undergoes cyclic loading such that \( \sigma_{\text{max}} = 60 \text{ kpsi} \) and \( \sigma_{\text{min}} = -20 \text{ kpsi} \). For the material, \( S_{ut} = 80 \text{ kpsi} \), \( S_y = 65 \text{ kpsi} \), a fully corrected endurance limit of \( S_e = 40 \text{ kpsi} \), and \( f = 0.9 \). Estimate the number of cycles to a fatigue failure using:

(a) Modified Goodman criterion.
(b) Gerber criterion.

Solution

From the given stresses,

\[
\sigma_a = \frac{60 - (-20)}{2} = 40 \text{ kpsi} \quad \sigma_m = \frac{60 + (-20)}{2} = 20 \text{ kpsi}
\]

(a) For the modified Goodman criterion, Eq. (6–46), the fatigue factor of safety based on infinite life is

\[
n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}} = \frac{1}{\frac{40}{40} + \frac{20}{80}} = 0.8
\]
This indicates a finite life is predicted. The S-N diagram is only applicable for completely reversed stresses. To estimate the finite life for a fluctuating stress, we will obtain an equivalent completely reversed stress that is expected to be as damaging as the fluctuating stress. A commonly used approach is to assume that since the modified Goodman line represents all stress situations with a constant life of $10^6$ cycles, other constant-life lines can be generated by passing a line through $(S_{ut}, 0)$ and a fluctuating stress point $(\sigma_m, \sigma_a)$. The point where this line intersects the $\sigma_a$ axis represents a completely reversed stress (since at this point $\sigma_m = 0$), which predicts the same life as the fluctuating stress.

This completely reversed stress can be obtained by replacing $S_e$ with $\sigma_{rev}$ in Eq. (6–46) for the modified Goodman line resulting in

$$
\sigma_{rev} = \frac{\sigma_a}{1 - \frac{\sigma_m}{S_{ut}}} = \frac{40}{1 - \frac{20}{80}} = 53.3 \text{ kpsi}
$$
Example 6–12 (continued)

From the material properties, Eqs. (6–14) to (6–16), p. 293, give

\[
a = \frac{(fS_{ut})^2}{S_e} = \frac{[0.9(80)]^2}{40} = 129.6 \text{ kpsi}
\]

\[
b = -\frac{1}{3} \log \left(\frac{fS_{ut}}{S_e}\right) = -\frac{1}{3} \log \left[\frac{0.9(80)}{40}\right] = -0.0851
\]

\[
N = \left(\frac{\sigma_{rev}}{a}\right)^{1/b} = \left(\frac{\sigma_{rev}}{129.6}\right)^{-1/0.0851}
\]

Substituting \(\sigma_{rev}\) into Eq. (1) yields

\[
N = \left(\frac{53.3}{129.6}\right)^{-1/0.0851} = 3.4(10^4) \text{ cycles}
\]

Answer
Example 6–12 (continued)

(b) For Gerber, similar to part (a), from Eq. (6–47),

$$\sigma_{rev} = \frac{\sigma_a}{1 - \left(\frac{\sigma_m}{S_{ut}}\right)^2} = \frac{40}{1 - \left(\frac{20}{80}\right)^2} = 42.7 \text{ kpsi}$$

Again, from Eq. (1),

$$N = \left(\frac{42.7}{129.6}\right)^{-1/0.0851} \approx 4.6(10^5) \text{ cycles} \quad \text{Answer}$$

Comparing the answers, we see a large difference in the results. Again, the modified Goodman criterion is conservative as compared to Gerber for which the moderate difference in $S_f$ is then magnified by a logarithmic $S, N$ relationship.