MAE 322
Machine Design
Shafts -2

Dr. Hodge Jenkins
Mercer University
Deflection Considerations

- Deflection analysis requires complete geometry & loading information for the entire shaft.
- Allowable deflections at components will depend on the component manufacturer’s specifications.
- Typical ranges are given in Table 7–2

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<th>Table 7–2: Typical Maximum Ranges for Slopes and Transverse Deflections</th>
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<td>Spur gears with $P &lt; 10$ teeth/in</td>
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<td>Spur gears with $11 &lt; P &lt; 19$</td>
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<td>Spur gears with $20 &lt; P &lt; 50$</td>
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Deflection Considerations

- Linear & angular deflections, should be checked at gears and bearings.

- **Deflection analysis** is straightforward, but very lengthy and tedious to carry out manually. Consequently, shaft deflection analysis is **almost always done with the assistance of software (usually FEA)**.

- For this reason, a common approach is to size critical locations for stress, then fill in reasonable size estimates for other locations, then check deflection using FEA or other software.

- Software options include specialized shaft software, general beam deflection software, and finite element analysis (FEA) software.
Angular Deflection of Shafts

- For stepped shaft with individual cylinder length $l_i$ and torque $T_i$, the angular deflection can be estimated from
  \[
  \theta = \sum \theta_i = \sum \frac{T_i l_i}{G_i J_i}
  \] (7–19)

- For constant torque throughout homogeneous material
  \[
  \theta = \frac{T}{G} \sum \frac{l_i}{J_i}
  \] (7–20)

- Experimental evidence shows that these equations slightly underestimate the angular deflection.

- Torsional stiffness of a stepped shaft is
  \[
  \frac{1}{k} = \sum \frac{1}{k_i}
  \] (7–21)
Critical Speeds for Shafts

• A shaft has a critical speed at which its deflections become unstable.
• Components attached to the shaft cause an even lower critical speed for the shaft.
• Designers should ensure that the lowest critical speed is at least twice the operating speed.

\[ \omega_1 = \sqrt{\frac{\text{Stiffness}}{\text{Inertia}}} = \sqrt{\frac{\text{Torsional stiffness}}{\text{Mass Moment of Inertia}}} \]

• \( \omega_1 \) is the fundamental (lowest) frequency
Critical Speeds for Shafts

- For a simply supported shaft of uniform diameter, the first critical speed is
  \[ \omega_1 = \left( \frac{\pi}{l} \right)^2 \sqrt{\frac{EI}{m}} = \left( \frac{\pi}{l} \right)^2 \sqrt{\frac{gEI}{Ay}} \]  
  \hspace{1cm} (7–22)

- For an ensemble of attachments, Rayleigh’s method for lumped masses gives
  \[ \omega_1 = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \]  
  \hspace{1cm} (7–23)
  
  - \( w_i \) is the weight of the \( i \)th location and \( y_i \) is the deflection at the \( i \)th body location

- Or Finite Element Model for modal analysis
Eq. (7–23) can be applied to the shaft itself by partitioning the shaft into segments.
Critical Speeds for Shafts

• The first critical speed can be approximated from Eq. (7–30) as

\[ \frac{1}{\omega_1^2} \gg \frac{1}{\omega_2^2}, \text{ and } \frac{1}{\omega_3^2} \]

• Extending this idea to an \( n \)-body shaft, we obtain Dunkerley’s equation,

\[ \frac{1}{\omega_1^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2} \]  \hspace{1cm} (7–31)

\[ \frac{1}{\omega_1^2} \approx \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2} \]  \hspace{1cm} (7–32)

Shigley’s Mechanical Engineering Design
Critical Speeds for Shafts

- Since Dunkerley’s method estimates a frequency below the actual fundamental frequency.
- Ralyeigh’s method converges from above.
- Thus, the two together can bound the actual fundamental frequency.
Example 7–5

Consider a simply supported steel shaft as depicted in Fig. 7–14, with 1 in diameter and a 31-in span between bearings, carrying two gears weighing 35 and 55 lbf.

(a) Find the influence coefficients.
(b) Find \( \sum wy \) and \( \sum wy^2 \) and the first critical speed using Rayleigh’s equation, Eq. (7–23).
(c) From the influence coefficients, find \( \omega_{11} \) and \( \omega_{22} \).
(d) Using Dunkerley’s equation, Eq. (7–32), estimate the first critical speed.

Fig. 7–14
(a) \[ I = \frac{\pi d^4}{64} = \frac{\pi (1)^4}{64} = 0.04909 \text{ in}^4 \]

\[ 6EI \ell = 6(30)10^6(0.04909)31 = 0.2739(10^9) \text{ lbf \cdot in}^3 \]

From Eq. set (7–24),

\[ \delta_{11} = \frac{24(7)(31^2 - 24^2 - 7^2)}{0.2739(10^9)} = 2.061(10^{-4}) \text{ in/lbf} \quad \text{Answer} \]

\[ \delta_{22} = \frac{11(20)(31^2 - 11^2 - 20^2)}{0.2739(10^9)} = 3.534(10^{-4}) \text{ in/lbf} \quad \text{Answer} \]

\[ \delta_{12} = \delta_{21} = \frac{11(7)(31^2 - 11^2 - 7^2)}{0.2739(10^9)} = 2.224(10^{-4}) \text{ in/lbf} \quad \text{Answer} \]

\[ y_1 = w_1 \delta_{11} + w_2 \delta_{12} = 35(2.061)10^{-4} + 55(2.224)10^{-4} = 0.01945 \text{ in} \]

\[ y_2 = w_1 \delta_{21} + w_2 \delta_{22} = 35(2.224)10^{-4} + 55(3.534)10^{-4} = 0.02722 \text{ in} \]
Example 7–5 (continued)

(b) \[
\sum w_i y_i = 35(0.01945) + 55(0.02722) = 2.178 \text{ lbf} \cdot \text{in}
\]
\[
\sum w_i y_i^2 = 35(0.01945)^2 + 55(0.02722)^2 = 0.05399 \text{ lbf} \cdot \text{in}^2
\]
\[
\omega = \sqrt{\frac{386.1(2.178)}{0.05399}} = 124.8 \text{ rad/s, or 1192 rev/min}
\]

(c) \[
\frac{1}{\omega_{11}^2} = \frac{w_1}{g} \delta_{11}
\]
\[
\omega_{11} = \sqrt{\frac{g}{w_1 \delta_{11}}} = \sqrt{\frac{386.1}{35(2.061)10^{-4}}} = 231.4 \text{ rad/s, or 2210 rev/min}
\]
\[
\omega_{22} = \sqrt{\frac{g}{w_2 \delta_{22}}} = \sqrt{\frac{386.1}{55(3.534)10^{-4}}} = 140.9 \text{ rad/s, or 1346 rev/min}
\]
Example 7–5 (continued)

\[
\frac{1}{\omega_1^2} \approx \sum \frac{1}{\omega_{ii}^2} = \frac{1}{231.4^2} + \frac{1}{140.9^2} = 6.905 \times 10^{-5}
\]

\[
\omega_1 \approx \sqrt{\frac{1}{6.905 \times 10^{-5}}} = 120.3 \text{ rad/s, or } 1149 \text{ rev/min}
\]

which is less than part \( b \), as expected.
From Eq. (7–33),

\[ w_{1c} = w_1 \frac{\delta_{11}}{\delta_{cc}} = 35 \frac{2.061 \times 10^{-4}}{4.215 \times 10^{-4}} = 17.11 \text{ lbf} \]

\[ w_{2c} = w_2 \frac{\delta_{22}}{\delta_{cc}} = 55 \frac{3.534 \times 10^{-4}}{4.215 \times 10^{-4}} = 46.11 \text{ lbf} \]

\[ \omega = \sqrt{\frac{g}{\delta_{cc} \sum w_{ic}}} = \sqrt{\frac{386.1}{4.215 \times 10^{-4} (17.11 + 46.11)}} = 120.4 \text{ rad/s, or 1150 rev/min} \]

which, except for rounding, agrees with part \( d \), as expected.

Fig. 7–14 (b)
(f) For the shaft, $E = 30(10^6)$ psi, $\gamma = 0.282$ lbf/in$^3$, and $A = \pi(1^2)/4 = 0.7854$ in$^2$. Considering the shaft alone, the critical speed, from Eq. (7–22), is

$$\omega_s = \left(\frac{\pi}{l}\right)\sqrt{\frac{gEI}{A\gamma}} = \left(\frac{\pi}{31}\right)\sqrt{\frac{386.1(30)10^6(0.04909)}{0.7854(0.282)}}$$

$$= 520.4 \text{ rad/s, or 4970 rev/min} \quad \text{Answer}$$

We can simply add $1/\omega_s^2$ to the right side of Dunkerley’s equation, Eq. (1), to include the shaft’s contribution,

$$\frac{1}{\omega_1^2} \approx \frac{1}{520.4^2} + 6.905(10^{-5}) = 7.274(10^{-5})$$

$$\omega_1 \approx 117.3 \text{ rad/s, or 1120 rev/min} \quad \text{Answer}$$

which is slightly less than part $d$, as expected.
Example 7–5 (continued)

The shaft’s first critical speed $\omega_s$ is just one more single effect to add to Dunkerley’s equation. Since it does not fit into the summation, it is usually written up front.

$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_s^2} + \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2}$$  \hspace{1cm} \text{Answer} \hspace{1cm} (7–34)

Common shafts are complicated by the stepped-cylinder geometry, which makes the influence-coefficient determination part of a numerical solution.