Beam Deflections
Moment Sign Convention

Positive internal moment concave upwards

Negative internal moment concave downwards
\( \nu \) is measured the deflection of the beam

\( \Theta \) is the slope for small angles
Differential equations of the deflection curve

- Beams with small angles of rotation, and small deflection
- The structures encountered in everyday life, such as buildings, automobiles, aircraft, ships undergo relatively small changes in shape while in service. Therefore, we assume **small angles** of rotation and **very small deflections**

\[
\frac{1}{\rho} \approx \frac{d^2 v}{dx^2}
\]

**Differential equation of the curve with radius, \( \rho \)**

\[
\frac{d^2 v}{dx^2} = \frac{M}{EI}
\]

**Differential equation of the deflection curve, \( v \)**

\( M \) is bending moment

\( EI \) is commonly called flexural rigidity
Differential equations of the deflection curve

\[
\frac{d^2 v}{dx^2} = \frac{M}{EI}
\]

Slope \( \sim \theta \)
(small \( \theta \))

\[
\frac{dv}{dx} = \int \frac{M}{EI} \, dx
\]

Deflection, \( v \)

\[
v = \int \left\{ \int \frac{M}{EI} \, dx \right\} \, dx
\]
Differential equations of the deflection curve

- Thus for prismatic beams

\[ EI \frac{d^2v}{dx^2} = M \]

BENDING – MOMENT EQUATION

\[ EI \frac{d^3v}{dx^3} = V \]

SHEAR – FORCE EQUATION

\[ EI \frac{d^4v}{dx^4} = -q \]

LOAD EQUATION
Moment and deflection curve
Two Main Choices in Deflection Determination

• Find moment as function of beam length, \( x \)
  – Integrate Moment twice

• Use superposition of previously solved problems
  – Tables of common loadings, for easy look up
  – Use linear superposition for combined loads, adding or subtracting as appropriate
## Simply Supported Beam Slopes and Deflections

<table>
<thead>
<tr>
<th>Beam</th>
<th>Slope</th>
<th>Deflection</th>
<th>Elastic Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Beam Diagram 1" /></td>
<td>$\theta_{\text{max}} = -\frac{PL^2}{16EI}$</td>
<td>$v_{\text{max}} = -\frac{PL^3}{48EI}$</td>
<td>$v = \frac{-Px}{48EI} (3L^2 - 4x^2)$, $0 \leq x \leq L/2$</td>
</tr>
<tr>
<td><img src="image2" alt="Beam Diagram 2" /></td>
<td>$\theta_1 = \frac{-Pab(L + b)}{6EI}$, $\theta_2 = \frac{Pab(L + a)}{6EI}$</td>
<td>$v \bigg</td>
<td>_{x=a} = \frac{-Pb\alpha}{6EI} (L^2 - b^2 - a^2)$</td>
</tr>
<tr>
<td><img src="image3" alt="Beam Diagram 3" /></td>
<td>$\theta_1 = \frac{-M_0L}{6EI}$, $\theta_2 = \frac{M_0L}{3EI}$</td>
<td>$v_{\text{max}} = \frac{-M_0L^3}{\sqrt{243EI}}$, at $x = 0.5774L$</td>
<td>$v = \frac{-M_0\alpha}{6EI} (L^2 - x^2)$</td>
</tr>
<tr>
<td><img src="image4" alt="Beam Diagram 4" /></td>
<td>$\theta_{\text{max}} = -\frac{wL^3}{24EI}$</td>
<td>$v_{\text{max}} = \frac{-5wL^4}{384EI}$</td>
<td>$v = \frac{-wx}{24EI} (x^3 - 2Lx^2 + L^2)$</td>
</tr>
<tr>
<td><img src="image5" alt="Beam Diagram 5" /></td>
<td>$\theta_1 = \frac{-3wL^3}{128EI}$, $\theta_2 = \frac{7wL^3}{384EI}$</td>
<td>$v \bigg</td>
<td>_{x=L/2} = \frac{-5wL^4}{768EI}$</td>
</tr>
<tr>
<td><img src="image6" alt="Beam Diagram 6" /></td>
<td></td>
<td>$v_{\text{max}} = -0.006563 \frac{wL^4}{EI}$, at $x = 0.4598L$</td>
<td>$v = \frac{-wL}{384EI} (8x^3 - 24Lx^2 + 17L^2 - x^3)$, $L/2 \leq x &lt; L$</td>
</tr>
<tr>
<td><img src="image7" alt="Beam Diagram 7" /></td>
<td>$\theta_1 = \frac{-7wL^3}{360EI}$, $\theta_2 = \frac{w_0L^3}{45EI}$</td>
<td>$v_{\text{max}} = -0.00652 \frac{w_0L^4}{EI}$, at $x = 0.5193L$</td>
<td>$v = \frac{-w_0x}{360EI} (3x^4 - 10L^2x^2 + 7L^4)$</td>
</tr>
</tbody>
</table>
### Cantilevered Beam Slopes and Deflections

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<tbody>
<tr>
<td><img src="image1.png" alt="Beam Diagram" /></td>
<td>$\theta_{\text{max}} = \frac{-PL^2}{2EI}$</td>
<td>$v_{\text{max}} = \frac{-PL^3}{3EI}$</td>
<td>$v = \frac{-Px^2}{6EI} (3L - x)$</td>
</tr>
<tr>
<td><img src="image2.png" alt="Beam Diagram" /></td>
<td>$\theta_{\text{max}} = \frac{-PL^2}{8EI}$</td>
<td>$v_{\text{max}} = \frac{-5PL^3}{48EI}$</td>
<td>$v = \frac{-Px^2}{6EI} \left( \frac{3}{2} L - x \right)$, $0 \leq x \leq L/2$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Beam Diagram" /></td>
<td>$\theta_{\text{max}} = \frac{-wL^3}{6EI}$</td>
<td>$v_{\text{max}} = \frac{-wL^4}{8EI}$</td>
<td>$v = \frac{-wx^2}{24EI} \left( x^2 - 4Lx + 6L^2 \right)$</td>
</tr>
<tr>
<td><img src="image4.png" alt="Beam Diagram" /></td>
<td>$\theta_{\text{max}} = \frac{M_0L}{EI}$</td>
<td>$v_{\text{max}} = \frac{M_0L^2}{2EI}$</td>
<td>$v = \frac{M_0x^2}{2EI}$</td>
</tr>
</tbody>
</table>
Example: Find deflection in middle
### Example solution: $1 + 2$

**Diagram 1:**

- $\theta_1 = \frac{-Pab(L + b)}{6EIL}$
- $\theta_2 = \frac{Pab(L + a)}{6EIL}$

- $v \bigg|_{x=a} = \frac{-Pba}{6EIL} (L^2 - b^2 - a^2)$

- $0 \leq x \leq a$

**Diagram 2:**

- $\theta_{\text{max}} = \frac{-wL^3}{24EI}$
- $v_{\text{max}} = \frac{-5wL^4}{384EI}$

- $v = \frac{-wx}{24EI} (x^3 - 2Lx^2 + L^3)$