Bode plots for 2$^{\text{nd}}$ Order systems
2\textsuperscript{nd} Order Systems

- Everything applies, except the break point
- Magnitude and Phase with $s=j\omega$

\[
G(s) = C_0 \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}
\]

“Double” Breakpoint at $\omega_n$
The asymptotic approaches described for real poles can be extended to systems with complex conjugate poles (and zeros).

\[ G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta \omega_n \omega + \omega_n^2} = \frac{1}{1 + 2\zeta \left( \frac{\omega}{\omega_n} \right) + \left( \frac{j \omega}{\omega_n} \right)^2} \] (Normalized)

\[ |G(j\cdot 0)| = 1 \]
\[ \angle G(j\cdot 0) = 0^\circ \]
\[ \lim_{\omega \to \infty} |G(j\omega)| = 0 \]
\[ \lim_{\omega \to \infty} \angle G(j\cdot 0) = -180^\circ \]
Polar Representation

In polar terms:

\[ |G(j\omega)| = \frac{1}{R_1R_2} \]
\[ \angle G(j\omega) = -(\theta_1 + \theta_2) \]

\[ \omega_n = \sqrt{x^2 + y^2} \]
\[ \zeta = \cos\left(\frac{y}{\omega_n}\right) \]

Note: As \( R_1 \) is reduced in size, \( |G(j\omega)| \) increases. For \( \zeta < 1/\sqrt{2} = 0.707 \), \( |G(j\omega)| \) will peak higher than DC gain at the resonance frequency

\[ \omega_R = \omega_n \sqrt{1 - 2\zeta^2} \]
Peak Frequency Gain

The peak value is given by: \[ M_R = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \]

Note: \[ M_R \left( \zeta = 1/\sqrt{2} \right) = \frac{1}{2 \cdot \frac{1}{\sqrt{2}} \sqrt{1 - \frac{1}{2}}} = \frac{1}{2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} = 1 \]

(a system with \( \zeta = 1/\sqrt{2} \) is termed maximally flat)

\[ \lim_{\zeta \to 0} M_R(\zeta) = \infty \]

Clearly since \( R_1 \to 0 \) and \( \frac{1}{R_1} \to \infty \)

\( \omega_R \) and \( M_R \) may be computed and the Bode plots may be sketched.

The Bode angle plot always starts off at 0° for a second order system, crosses at -90° and asymptotically approaches -180°.
Lightly Damped Systems

The lower the $\zeta$, the sharper the peak on the magnitude plot and the steeper the curve on the angle plot.

At $\zeta = 0$ the peak is infinite on the magnitude plot and the phase shift drops vertically from $0^\circ$ to $-180^\circ$. 

[(Diagram of complex plane with points (x + yj) and (x - yj) and angles $\theta_1$ and $\theta_2$)]
2nd Order System Bode Plots, ($\zeta = 0, .1, ..., 1$)
2\textsuperscript{nd} Order Systems

- Everything applies, except the break point
- Magnitude and Phase with $s = j\omega$

$$G(s) = C_0 \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

“Double” Breakpoint at $\omega_n$
Bode Plots Approximations

• Because Double break point at $\omega_n$
• If in $s^2+2\xi\omega_n s+\omega_n^2$ in denominator
  • -40 db/decade in denominator at $\omega_n$
  • -180 deg shift (starting a decade below, to decade above at $\omega_n$)
  • -90 deg at break point at $\omega_n$
\[ G(s) = \frac{9}{s^2 + 2s + 9} \]
$$G(s) = \frac{9}{s^3 + 2s^2 + 9s}$$
$$G(s) = \frac{(s+1)}{(s^2+2s+100)}$$